UIUC UNDERGRADUATE MATH CONTEST
APRIL 13, 2002, 10 am – 1 pm
SOLUTIONS

Problem 1

Without any numerical calculations, determine which of the two numbers e^π and π^e is larger.

Solution. Let $a = e^\pi$ and $b = \pi^e$. We will show that $a > b$. Since $\ln a = \pi \ln e = \pi$ and $\ln b = e \ln \pi$, and since taking logarithms preserves inequalities, we see that $a > b$ holds if and only if $(*)$ $(\ln e)/e > (\ln \pi)/\pi$. Now consider the function $f(x) = (\ln x)/x$. We have $f'(x) = (1 - \ln x)/x^2$, so f is decreasing for $x > e$, and since $\pi > e$, it follows that $f(\pi) < f(e)$ which is equivalent to $(*)$.

Problem 2

Let $0ABC$ be a tetrahedron with three right angles at the point O. Let S_A be the area of the face opposite to the point A, i.e., the area of the triangle OBC, and define S_B, S_C, and S_O analogously. Prove that $S_O^2 = S_A^2 + S_B^2 + S_C^2$.

Solution. Place the tetrahedron so that its vertices are located at $O = (0,0,0)$, $A = (a,0,0)$, $B = (0,b,0)$, and $C = (0,0,c)$. Clearly $S_A = bc/2$, $S_B = ac/2$, and $S_C = ab/2$. Moreover, the area S_O of the triangle ABC is $1/2$ times the area of the parallelogram determined by the vectors $AB = (-a,b,0)$ and $AC = (-a,0,c)$, which in turn is given by the magnitude of the cross product of these two vectors. Computing this cross product gives (bc, ac, ab), so

$$S_0^2 = (1/2)^2 \|(bc, ac, ab)\|^2 = \frac{1}{4}\left((bc)^2 + (ac)^2 + (ab)^2\right) = S_A^2 + S_B^2 + S_C^2.$$

Problem 3

Let $\theta_n = \arctan n$. Prove that, for $n = 1, 2, \ldots$,

$$\theta_{n+1} - \theta_n < \frac{1}{n^2 + n}.$$

Solution. Using the fact that $\arctan x$ has derivative $1/(1+x^2)$, we obtain

$$\theta_{n+1} - \theta_n = \arctan(n+1) - \arctan n = \int_n^{n+1} \frac{dx}{1+x^2}$$
$$< \int_n^{n+1} \frac{dx}{x^2} = \frac{1}{n} - \frac{1}{n+1} = \frac{1}{n(n+1)}.$$

Problem 4

Determine, with proof, whether the (double) series

$$\sum_{(*)} \left(\frac{m}{n}\right)^{mn},$$

taken over all pairs (m, n) of positive integers satisfying

$$(*) \qquad n = 2, 3, \ldots, \quad m = 1, 2, \ldots, n-1$$

converges.

Solution. Split the range $(*)$ into the subranges (I) $n = 2, 3, \ldots, m \leq n/2$ and (II) $n = 2, 3, \ldots, n/2 < m \leq n-1$. It suffices to show that the series taken over each of these two ranges converge.

In the range (I), we have $(m/n)^{mn} \leq (1/2)^{mn}$. Summing this upper bound first over n (from $n = 2m$ to infinity) gives a geometric series with sum $(1/2)^{m(2m)}/(1-(1/2)^m)$ which is at most $(1/2)^m$ (with a lot to spare). Since $\sum_{n=1}^{\infty}(1/2)^m$ converges, so does the series $(*)$ over the subrange (I). To deal with the second subrange, we set $h = n - m$, so that $1 \leq h < n/2$ in the range (II). Using the bounds $(m/n) = (1 - h/n) \leq e^{-h/n}$ and $mn \geq n^2/2$, we obtain $(m/n)^{mn} \leq \exp\{-\frac{h}{n} \cdot \frac{1}{2}n^2\} = \exp\{-\frac{hn}{2}\}$. Summing the last term over n, from $n = 2h$ to infinity, gives again a geometric series, with sum $e^{-h^2}/(1-e^{-h/2})$ which is at most $e^{-h}(1-e^{-1/2})^{-1}$. Summing the latter bound, from $h = 1$ to infinity, we again obtain a convergent geometric series. Hence the series over the subrange (II) converges as well.

Problem 5

Let $a_1 = 2$, $a_2 = 4$, $a_3 = 8$, and for $n \geq 4$ define a_n to be last digit of the sum of the preceding **three** terms in the sequence. Thus the first few terms of this sequence of digits are (in concatenated form) $248468828\ldots$. Determine, with proof, whether or not the string 2002 occurs somewhere in this sequence.

Solution. First note that the sequence can be continued backwards in a unique manner by setting $a_{n-1} = a_{n+2} - a_{n+1} - a_n \mod 10$. Doing so, one finds that the first four terms prior to the given terms are 2, 0, 0, and 2. Thus, the string 2002 occurs in the extended sequence. To show that it also occurs in the original sequence (i.e., to the right of $2484\ldots$), note that the sequence is uniquely determined, backwards and forwards, by any three consecutive digits in the sequence. Since there are finitely many possibilities for such triples of consecutive digits, one such triple must occur again in the sequence, and the sequence is therefore is periodic (in both directions). In particular, any string that occurs somewhere in the extended sequence, occurs infinitely often and arbitrarily far out along the given sequence. Hence 2002 does occur in this sequence.

Problem 6

Call a set of integers A double-free if it does not contain two elements a and a' with $a' = 2a$. Determine, with proof, the size of the largest double-free subset of the set $\{1, 2, \ldots, 256\}$.

Solution. We will show that the maximal cardinality sought is 171. To prove that the cardinality cannot exceed 171, suppose $A \subset \{1, 2, \ldots, 256\}$ is double-free. Given any element $a \in A$, let a_0 denote the odd part of a, so that $a = a_0 2^i$ with a_0 odd and i a nonnegative integer. For each odd integer m, let A_m denote the set of $a \in A$ with $a_0 = m$. The sets A_m, $m = 1, 3, \ldots, 255$ partition the set A, so $|A| = |A_1| + |A_3| + \cdots + |A_{255}|$. To obtain an upper bound for $|A|$ we consider $|A_m|$ for different ranges of m.

If (1) $128 < m \leq 256$, then there can be at most one $a \in A$ with $a_0 = m$, namely $a = m$. Thus, the sum over $|A_m|$ for m in the range (1) is at most equal to the number of odd m in this range, i.e., 64.

If (2) $64 < m \leq 128$, then any $a \leq 256$ with $a_0 = m$ must be of the form $a = m$ or $a = 2m$, but because of the double-free condition at most one of these can belong to a. Hence $|A_m| \leq 1$ for m in the range (2), and the sum of $|A_m|$ over such m is at most 32.

If (3) $32 < m \leq 64$, then $a_0 = m$ implies that $a = m2^i$ with $i = 0$, 1, or 2, but the double-free condition again implies that at most two of these can belong to A. Hence $|A_m| \leq 2$ in the range (3), and the sum of $|A_m|$ over m in this range is at most $16 \cdot 2 = 32$.

Similarly, considering the ranges (4) $16 < m \leq 32$, (5) $8 < m \leq 16$, (6) $4 < m \leq 8$, (7) $2 < m \leq 4$ (i.e., $m = 3$), and (8) $m = 1$, we see that $|A_m|$ is at most 2 in the range (4), 3 in the ranges (5) and (6), 4 in the range (7), and 5 in the range 8, and the corresponding sums over $|A_m|$ are bounded by $8 \cdot 2 = 16$, $4 \cdot 3 = 12$, $2 \cdot 3 = 6$, $1 \cdot 4 = 4$, and $1 \cdot 5 = 5$, respectively. Adding up these bounds, we obtain

$$|A| \leq 64 + 32 + 32 + 16 + 12 + 6 + 4 + 5 = 171.$$

To show that this bound can be achieved, take A to be the set of integers $n \leq 256$ that are of the form $a_0 2^i$ with a_0 odd and $i = 0, 2, \ldots$. In this case, it is easy to check that the inequalities for $|A_m|$ in the above argument become equalities, and so we have $|A| = 171$.