UIUC UNDERGRADUATE MATH CONTEST

APRIL 13, 2002, 10 am – 1 pm

Problem 1

Without any numerical calculations, determine which of the two numbers e^π and π^e is larger.

Problem 2

Let $0ABC$ be a tetrahedron with three right angles at the point O. Let S_A be the area of the face opposite to the point A, i.e., the area of the triangle OBC, and define S_B, S_C, and S_O analogously. Prove that $S_O^2 = S_A^2 + S_B^2 + S_C^2$.

Problem 3

Let $\theta_n = \arctan n$. Prove that, for $n = 1, 2, \ldots$,

$$\theta_{n+1} - \theta_n < \frac{1}{n^2 + n}.$$

Problem 4

Determine, with proof, whether the (double) series

$$\sum_{(*)} \left(\frac{m}{n}\right)^{mn},$$

taken over all pairs (m, n) of positive integers satisfying

$$(*) \qquad n = 2, 3, \ldots, \quad m = 1, 2, \ldots, n-1$$

converges.

Problem 5

Let $a_1 = 2$, $a_2 = 4$, $a_3 = 8$, and for $n \geq 4$ define a_n to be last digit of the sum of the preceding **three** terms in the sequence. Thus the first few terms of this sequence of digits are (in concatenated form) $248468828\ldots$. Determine, with proof, whether or not the string 2002 occurs somewhere in this sequence.

Problem 6

Call a set of integers A double-free if it does not contain two elements a and a' with $a' = 2a$. Determine, with proof, the size of the largest double-free subset of the set $\{1, 2, \ldots, 256\}$.