UIUC UNDERGRADUATE MATH CONTEST
APRIL 21, 2001, 10 am – 1 pm
SOLUTIONS

Problem 1

Given a positive integer n, let n_1 be the sum of digits (in decimal) of n, n_2 the sum of digits of n_1, n_3 the sum of digits of n_2, etc. The sequence $\{n_i\}$ eventually becomes constant, and equal to a single digit number. Call this number $f(n)$. For example, $f(1999) = 1$ since for $n = 1999$, $n_1 = 28$, $n_2 = 10$, $n_3 = n_4 = \cdots = 1$. How many positive integers $n \leq 2001$ are there for which $f(n) = 9$?

Solution. Since an integer is divisible by 9 if and only if its sum of digits is divisible by 9, the numbers n with $f(n) = 9$ are exactly the multiples of 9. Since $2001 = 9 \cdot 222 + 3$, there are 222 such numbers below 2001.

Problem 2

Let x, y, and z be nonzero real numbers satisfying

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{1}{x+y+z}.$$

Show that $x^n + y^n + z^n = (x+y+z)^n$ for any odd integer n.

Solution. From the given relation one obtains, after clearing denominators and simplifying, $(x+y)(x+z)(y+z) = 0$. Hence $x = -y$, $x = -z$, or $y = -z$. In the first case, $x^n + y^n = 0$ for odd n, and so $x^n + y^n + z^n = (x+y+z)^n$. The other cases are analogous.

Problem 3

Suppose that an equilateral triangle is given in the plane, with none of its sides vertical. Let m_1, m_2, m_3 denote the slopes of the three sides. Show that

$$m_1 m_2 + m_2 m_3 + m_3 m_1 = -3.$$

Solution. Let A, B, and C be the vertices of the triangle, labelled so that the path $ABCA$ is counter-clockwise, and let m_1, m_2, m_3 denote the slopes of the three sides AB, BC, and AC, respectively. Without loss of generality, we may assume that the vertex A is located at the origin. Then $m_1 = \tan\theta$, $m_2 = \tan(\theta + 2\pi/3)$, and $m_3 = \tan(\theta + \pi/3)$, where θ is the angle between the positive x axis and AB. (Note that $\theta \geq 0$ if the triangle lies entirely in the first quadrant; $\theta < 0$ if the triangle extends into the fourth quadrant.) Using the identity $\tan(x+y) = (\tan x + \tan y)/(1 - \tan x \tan y)$, we get, with $T = \tan\theta$,

$$m_1 m_2 = \frac{T(T-\sqrt{3})}{1+T\sqrt{3}}, \quad m_2 m_3 = \frac{(T-\sqrt{3})}{1+T\sqrt{3}} \cdot \frac{(T+\sqrt{3})}{1-T\sqrt{3}}, \quad m_3 m_1 = \frac{T(T+\sqrt{3})}{1-T\sqrt{3}}.$$

Adding these three terms and simplifying gives $m_1m_2+m_2m_3+m_3m_1 = -3$, independently of the value of T (and θ).

Problem 4

Let $x_1 \geq x_2 \geq \cdots \geq x_n > 0$ be real numbers. Prove that

$$\frac{x_1}{x_2} + \frac{x_2}{x_3} + \cdots + \frac{x_{n-1}}{x_n} + \frac{x_n}{x_1} \leq \frac{x_2}{x_1} + \frac{x_3}{x_2} + \cdots + \frac{x_n}{x_{n-1}} + \frac{x_1}{x_n}.$$

Solution. Set $q_i = x_i/x_{i+1}$. Then $x_1/x_n = \prod_{i=1}^{n-1} q_i$, so the inequality to be proved can be written as

$$\sum_{i=1}^{n-1} q_i + \prod_{i=1}^{n-1} \frac{1}{q_i} \leq \sum_{i=1}^{n-1} \frac{1}{q_i} + \prod_{i=1}^{n-1} q_i,$$

or equivalently

$$(*) \qquad \sum_{i=1}^{n-1} \left(q_i - \frac{1}{q_i} \right) - \prod_{i=1}^{n-1} q_i + \prod_{i=1}^{n-1} \frac{1}{q_i} \leq 0.$$

Let $f(q_1, \ldots, q_{n-1})$ denote the function on the left of $(*)$. The hypothesis that the x_i are non-increasing implies that $q_i \geq 1$ for all i. Since $f(1, \ldots, 1) = 0$, to prove $(*)$ it therefore suffices to show that the partial derivatives of f are ≤ 0 when $q_i \geq 1$ for all i. This is indeed the case: we have

$$\begin{aligned} \frac{\partial f}{\partial q_i} &= 1 + \frac{1}{q_i^2} - \prod_{j \neq i} q_j - \frac{1}{q_i^2} \prod_{j \neq i} \frac{1}{q_j} \\ &= \left(1 + \frac{1}{q_i^2} \right) \left(1 - \prod_{j \neq i} q_j \right) \leq 0, \end{aligned}$$

since $\prod_{j \neq i} q_j \geq 1$. (When $n = 2$, this last product is empty, but in that case the sums and products on the left of $(*)$ reduce to a single term (corresponding to $i = 1$), and a direct computation shows that the derivative of f with respect to q_1 is equal to zero, so the last inequality remains valid for this case.)

Problem 5

Suppose that $q(x)$ is a polynomial satisfying the differential equation

$$7 \frac{d}{dx} \{x q(x)\} = 3q(x) + 4q(x+1), \quad -\infty < x < \infty.$$

Show that $q(x)$ is necessarily a constant.

Solution. The left-hand side of the given equation is $7xq'(x) + 7q(x)$, so the equation simplifies to

$$(*) \qquad 7xq'(x) = -4q(x) + 4q(x+1) = 4\int_x^{x+1} q'(t)dt.$$

The left-hand side of $(*)$ is zero at $x = 0$, so by the mean value theorem for integrals (which can be applied here since $q(x)$ is a polynomial and hence has continuous derivatives of all orders) there exists a number $x_1 \in (0,1)$ with $q'(x_1) = 0$. Setting $x = x_1$ in $(*)$, we obtain a number $x_2 \in (x_1, x_1 + 1)$ with $q'(x_2) = 0$. Repeating this process, we obtain an infinite sequence $x_1 < x_2 < \cdots$ of values x at which $q'(x) = 0$. Since q' is a polynomial, q' must be identically zero. Hence q is constant.

Alternative solution: The above solution was the one we had in mind when posing the problem. However, all students who correctly solved the problem, did so via the following approach (or a variant of it): Write $q(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$, where $a_n \neq 0$. Then the left side of the differential equation is a polynomial of degree n with leading term $7a_n(n+1)x^n$, while the right-hand side has leading term $7a_n x^n$. Equating the coefficients of those terms, we obtain $7a_n(n+1) = 7a_n$; since $a_n \neq 0$, this can only hold when $n = 0$, i.e., when $q(x)$ is constant.

Problem 6

Evaluate the sum $\sum_{k=n}^{2n} \binom{k}{n} 2^{-k}$.

Solution. Let $S(n)$ denote the given sum. We claim that $S(n) = 1$ for all n. Since $S(1) = 1$, it suffices to show that $S(n+1) = S(n)$ for all n. Writing $k = n + 1 + h$ and using the identity $\binom{n+1+h}{n+1} = \binom{n+1+h}{h} = \binom{n+h}{h} + \binom{n+h}{h-1}$ for $h \geq 1$, we have

$$\begin{aligned}
2^{n+1}S(n+1) &= \sum_{h=0}^{n+1} \binom{n+1+h}{h} 2^{-h} = \sum_{h=0}^{n+1} \binom{n+h}{h} 2^{-h} + \sum_{h=1}^{n+1} \binom{n+h}{h-1} 2^{-h} \\
&= \sum_{h=0}^{n} \binom{n+h}{h} 2^{-h} + \binom{2n+1}{n+1} 2^{-n-1} + \sum_{h=0}^{n+1} \binom{n+1+h}{h} 2^{-h-1} - \binom{2n+2}{n+1} 2^{-n-2} \\
&= 2^n S(n) + 2^n S(n+1) + \left(\binom{2n+1}{n+1} - \frac{1}{2} \binom{2n+2}{n+1} \right) 2^{-n-1}.
\end{aligned}$$

Since $\binom{2n+2}{n+1} = \binom{2n+1}{n+1} + \binom{2n+1}{n} = 2\binom{2n+1}{n+1}$, the last term is zero, so we have $2^{n+1}S(n+1) = 2^n S(n) + 2^n S(n+1)$, and hence $S(n+1) = S(n)$, as claimed.