Problem 1

Given a positive integer \( n \), let \( n_1 \) be the sum of digits (in decimal) of \( n \), \( n_2 \) the sum of digits of \( n_1 \), \( n_3 \) the sum of digits of \( n_2 \), etc. The sequence \( \{n_i\} \) eventually becomes constant, and equal to a single digit number. Call this number \( f(n) \). For example, \( f(1999) = 1 \) since for \( n = 1999 \), \( n_1 = 28 \), \( n_2 = 10 \), \( n_3 = n_4 = \cdots = 1 \). How many positive integers \( n \leq 2001 \) are there for which \( f(n) = 9 \)?

Solution. Since an integer is divisible by 9 if and only if its sum of digits is divisible by 9, the numbers \( n \) with \( f(n) = 9 \) are exactly the multiples of 9. Since \( 2001 = 9 \cdot 222 + 3 \), there are 222 such numbers below 2001.

Problem 2

Let \( x, y, \) and \( z \) be nonzero real numbers satisfying

\[
\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{1}{x+y+z}.
\]

Show that \( x^n + y^n + z^n = (x+y+z)^n \) for any odd integer \( n \).

Solution. From the given relation one obtains, after clearing denominators and simplifying, \((x+y)(x+z)(y+z) = 0\). Hence \( x = -y, x = -z, \) or \( y = -z \). In the first case, \( x^n + y^n = 0 \) for odd \( n \), and so \( x^n + y^n + z^n = (x+y+z)^n \). The other cases are analogous.

Problem 3

Suppose that an equilateral triangle is given in the plane, with none of its sides vertical. Let \( m_1, m_2, m_3 \) denote the slopes of the three sides. Show that

\[
m_1m_2 + m_2m_3 + m_3m_1 = -3.
\]

Solution. Let \( A, B, \) and \( C \) be the vertices of the triangle, labelled so that the path \( ABCA \) is counter-clockwise, and let \( m_1, m_2, m_3 \) denote the slopes of the three sides \( AB, BC, \) and \( AC, \) respectively. Without loss of generality, we may assume that the vertex \( A \) is located at the origin. Then \( m_1 = \tan \theta, m_2 = \tan(\theta + 2\pi/3), \) and \( m_3 = \tan(\theta + \pi/3), \) where \( \theta \) is the angle between the positive \( x \) axis and \( AB \). (Note that \( \theta \geq 0 \) if the triangle lies entirely in the first quadrant; \( \theta < 0 \) if the triangle extends into the fourth quadrant.) Using the identity \( \tan(x + y) = (\tan x + \tan y)/(1 - \tan x \tan y) \), we get, with \( T = \tan \theta, \)

\[
m_1m_2 = \frac{T(T - \sqrt{3})}{1 + T\sqrt{3}}, \quad m_2m_3 = \frac{(T - \sqrt{3})(T + \sqrt{3})}{1 - T\sqrt{3}}, \quad m_3m_1 = \frac{T(T + \sqrt{3})}{1 - T\sqrt{3}}.
\]
Adding these three terms and simplifying gives \( m_1 m_2 + m_2 m_3 + m_3 m_1 = -3 \), independently of the value of \( T \) (and \( \theta \)).

**Problem 4**

Let \( x_1 \geq x_2 \geq \cdots \geq x_n > 0 \) be real numbers. Prove that

\[
\frac{x_1}{x_2} + \frac{x_2}{x_3} + \cdots + \frac{x_{n-1}}{x_n} + \frac{x_n}{x_1} \leq \frac{x_2}{x_1} + \frac{x_3}{x_2} + \cdots + \frac{x_n}{x_{n-1}} + \frac{x_1}{x_n}.
\]

**Solution.** Set \( q_i = x_i/x_{i+1} \). Then \( x_1/x_n = \prod_{i=1}^{n-1} q_i \), so the inequality to be proved can be written as

\[
\sum_{i=1}^{n-1} q_i + \prod_{i=1}^{n-1} \frac{1}{q_i} \leq \sum_{i=1}^{n-1} \frac{1}{q_i} + \prod_{i=1}^{n-1} q_i,
\]

or equivalently

\[
(*) \quad \sum_{i=1}^{n-1} \left( q_i - \frac{1}{q_i} \right) - \prod_{i=1}^{n-1} q_i + \prod_{i=1}^{n-1} \frac{1}{q_i} \leq 0.
\]

Let \( f(q_1, \ldots, q_{n-1}) \) denote the function on the left of \((*)\). The hypothesis that the \( x_i \) are non-increasing implies that \( q_i \geq 1 \) for all \( i \). Since \( f(1, \ldots, 1) = 0 \), to prove \((*)\) it therefore suffices to show that the partial derivatives of \( f \) are \( \leq 0 \) when \( q_i \geq 1 \) for all \( i \). This is indeed the case: we have

\[
\frac{\partial f}{\partial q_i} = 1 + \frac{1}{q_i^2} - \prod_{j \neq i} q_j - \frac{1}{q_i^2} \prod_{j \neq i} q_j
\]

\[
= \left( 1 + \frac{1}{q_i^2} \right) \left( 1 - \prod_{j \neq i} q_j \right) \leq 0,
\]

since \( \prod_{j \neq i} q_j \geq 1 \). (When \( n = 2 \), this last product is empty, but in that case the sums and products on the left of \((*)\) reduce to a single term (corresponding to \( i = 1 \)), and a direct computation shows that the derivative of \( f \) with respect to \( q_1 \) is equal to zero, so the last inequality remains valid for this case.)

**Problem 5**

Suppose that \( q(x) \) is a polynomial satisfying the differential equation

\[
7 \frac{d}{dx} \{ xq(x) \} = 3q(x) + 4q(x + 1), \quad -\infty < x < \infty.
\]

Show that \( q(x) \) is necessarily a constant.
**Solution.** The left-hand side of the given equation is $7xq'(x) + 7q(x)$, so the equation simplifies to

\[(*) \quad 7xq'(x) = -4q(x) + 4q(x + 1) = 4 \int_x^{x+1} q'(t)dt.\]

The left-hand side of (*) is zero at $x = 0$, so by the mean value theorem for integrals (which can be applied here since $q(x)$ is a polynomial and hence has continuous derivatives of all orders) there exists a number $x_1 \in (0,1)$ with $q'(x_1) = 0$. Setting $x = x_1$ in (*), we obtain a number $x_2 \in (x_1, x_1 + 1)$ with $q'(x_2) = 0$. Repeating this process, we obtain an infinite sequence $x_1 < x_2 < \cdots$ of values $x$ at which $q'(x) = 0$. Since $q'$ is a polynomial, $q'$ must be identically zero. Hence $q$ is constant.

**Alternative solution:** The above solution was the one we had in mind when posing the problem. However, all students who correctly solved the problem, did so via the following approach (or a variant of it): Write $q(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_0$, where $a_n \neq 0$. Then the left side of the differential equation is a polynomial of degree $n$ with leading term $7a_n(n+1)x^n$, while the right-hand side has leading term $7a_nx^n$. Equating the coefficients of those terms, we obtain $7a_n(n+1) = 7a_n$; since $a_n \neq 0$, this can only hold when $n = 0$, i.e., when $q(x)$ is constant.

**Problem 6**

Evaluate the sum $\sum_{k=0}^{2n} \binom{k}{n} 2^{-k}$.

**Solution.** Let $S(n)$ denote the given sum. We claim that $S(n) = 1$ for all $n$. Since $S(1) = 1$, it suffices to show that $S(n + 1) = S(n)$ for all $n$. Writing $k = n + 1 + h$ and using the identity $\binom{n+1+h}{n+1} = \binom{n+h}{h} + \binom{n+h}{h-1}$ for $h \geq 1$, we have

\[
2^{n+1}S(n+1) = \sum_{h=0}^{n+1} \binom{n+1+h}{h} 2^{-h} = \sum_{h=0}^{n+1} \binom{n+h}{h} 2^{-h} + \sum_{h=1}^{n+1} \binom{n+h}{h-1} 2^{-h}.
\]

\[
= \sum_{h=0}^{n} \binom{n+h}{h} 2^{-h} + \left(\binom{2n+1}{n+1} - \binom{2n+1}{n} + \sum_{h=0}^{n+1} \binom{n+1+h}{h} 2^{-h-1} - \binom{2n+2}{n+1} 2^{-n-2}\right) 2^{n-1}.
\]

Since $\binom{2n+2}{n+1} = \binom{2n+1}{n+1} + \binom{2n+1}{n}$, the last term is zero, so we have $2^{n+1}S(n+1) = 2^nS(n) + 2^nS(n+1)$, and hence $S(n+1) = S(n)$, as claimed.