UIUC UNDERGRADUATE MATH CONTEST

APRIL 21, 2001, 10 am – 1 pm

Problem 1

Given a positive integer n, let n_1 be the sum of digits (in decimal) of n, n_2 the sum of digits of n_1, n_3 the sum of digits of n_2, etc. The sequence $\{n_i\}$ eventually becomes constant, and equal to a single digit number. Call this number $f(n)$. For example, $f(1999) = 1$ since for $n = 1999$, $n_1 = 28$, $n_2 = 10$, $n_3 = n_4 = \cdots = 1$. How many positive integers $n \leq 2001$ are there for which $f(n) = 9$?

Problem 2

Let x, y, and z be nonzero real numbers satisfying

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{1}{x+y+z}.$$

Show that $x^n + y^n + z^n = (x+y+z)^n$ for any odd integer n.

Problem 3

Suppose that an equilateral triangle is given in the plane, with none of its sides vertical. Let m_1, m_2, m_3 denote the slopes of the three sides. Show that

$$m_1 m_2 + m_2 m_3 + m_3 m_1 = -3.$$

Problem 4

Let $x_1 \geq x_2 \geq \cdots \geq x_n > 0$ be real numbers. Prove that

$$\frac{x_1}{x_2} + \frac{x_2}{x_3} + \cdots + \frac{x_{n-1}}{x_n} + \frac{x_n}{x_1} \leq \frac{x_2}{x_1} + \frac{x_3}{x_2} + \cdots + \frac{x_n}{x_{n-1}} + \frac{x_1}{x_n}.$$

Problem 5

Suppose that $q(x)$ is a polynomial satisfying the differential equation

$$7\frac{d}{dx}\{xq(x)\} = 3q(x) + 4q(x+1), \quad -\infty < x < \infty.$$

Show that $q(x)$ is necessarily a constant.

Problem 6

Evaluate the sum $\sum_{k=n}^{2n} \binom{k}{n} 2^{-k}$.