UIUC UNDERGRADUATE MATH CONTEST

APRIL 15, 2000, 10 am – 1 pm

SOLUTIONS

Problem 1

Suppose that a_1, a_2, ..., a_n are n given integers. Prove that there exist integers r and s with $0 \le r < s \le n$ such that $a_{r+1} + a_{r+2} + \ldots + a_s$ is divisible by n.

Solution. Let $S_0 = 0$ and for $m = 1, 2, \ldots, n$, let $S_n = a_1 + a_2 + \cdots a_n$. By the pigeonhole principle, two of these $n + 1$ integers, say S_r and S_s (with $0 \le r < s \le n$), must leave the same remainder upon division by n. Hence $S_s - S_r = a_{r+1} + a_{r+2} + \cdots a_s$ is a multiple of n.

Problem 2

Let p be a point inside a triangle having sides of lengths a, b, c. Let h_a be the distance from p to the side of length a, and let h_b and h_c be defined analogously. Let $h := \min(h_a, h_b, h_c)$ and $s := (a + b + c)/2$. Prove that

$$h \le \sqrt{(s-a)(s-b)(s-c)/s}.$$

Solution. We study the area of the triangle. Draw lines from p to the three vertices. The areas of the resulting three new triangles are $(1/2)ah_a$, $(1/2)bh_b$, $(1/2)ch_c$. Thus the area of the original triangle is *at least*

$$\frac{1}{2}(a + b + c)\min(h_a, h_b, h_c) = sh.$$

On the other hand, by Heron's formula, the area of the original triangle is

$$\sqrt{s(s-a)(s-b)(s-c)}.$$

Thus, $sh \le \sqrt{s(s-a)(s-b)(s-c)}$, which is equivalent to the asserted inequality.

Problem 3

Let f and g be twice continuously differentiable functions on $[0, 1]$ with $f(0) = g(0) = 0 = f(1) = g(1)$. Suppose that $0 < f(x) < g(x)$ for $0 < x < 1$ and that $f''(x) < 0$ for $0 < x < 1$. Show that

$$\int_0^1 f'(x)^2\,dx \le \int_0^1 g'(x)^2\,dx.$$

Solution. Integrating by parts and using the assumptions $f(1) = g(1) = f(0) = g(0) = 0$, $f'' < 0$ and $f < g$, we have

$$\int_0^1 f'(x)^2 dx = f(x)f'(x)\Big|_0^1 - \int_0^1 f(x)f''(x)dx < -\int_0^1 g(x)f''(x)dx$$
$$= -g(x)f'(x)\Big|_0^1 + \int_0^1 g'(x)f'(x)dx = \int_0^1 g'(x)f'(x)dx$$

Also, by Cauchy's inequality,

$$\left(\int_0^1 g'(x)f'(x)dx\right)^2 \leq \left(\int_0^1 g'(x)^2 dx\right)\left(\int_0^1 f'(x)^2 dx\right),$$

Thus,

$$\left(\int_0^1 f'(x)^2 dx\right)^2 \leq \left(\int_0^1 g'(x)^2 dx\right)\left(\int_0^1 f'(x)^2 dx\right),$$

and dividing by $\int_0^1 f'(x)^2 dx$ gives the result.

Problem 4

Prove that if a, b, and c are odd positive integers, then the polynomial $ax^2 + bx + c$ has no rational roots.

Solution. Suppose that $x = r/s$ is a rational solution in lowest terms of the polynomial equation $ax^2 + bx + c = 0$. Then $(*)$ $ar^2 + brs + cs^2 = 0$. Since the fraction r/s is in lowest terms, at most one of r and s can be even. If exactly one of r and s is even and the other is odd, then two of the three terms on the left of $(*)$ are even and the third term is odd, so their sum cannot equal 0. If both r and s are odd, then all three terms on the left of $(*)$ are odd, and their sum again cannot be equal to 0. Thus, the equation $ax^2 + bx + c = 0$ does not have a rational root.

Problem 5

Evaluate the infinite series

$$\frac{1}{2^1 - 2^{-1}} + \frac{1}{2^2 - 2^{-2}} + \frac{1}{2^4 - 2^{-4}} + \frac{1}{2^8 - 2^{-8}} + \cdots$$

Solution. Let $f(x) = \sum_{n=0}^{\infty}(x^{-2^n} - x^{2^n})^{-1}$, so that the given series is $f(1/2)$. Writing each term in this series as a product $x^{2^n}(1 - x^{2^{n+1}})^{-1}$ and expanding the second factor into a geometric series, we get

$$f(x) = \sum_{n=0}^{\infty} x^{2^n} \sum_{k=0}^{\infty} x^{2^{n+1}k} = \sum_{n,k=0}^{\infty} x^{2^n + 2^{n+1}k} = \sum_{n,k=0}^{\infty} x^{2^n(2k+1)}.$$

In the last series, the exponents $2^n(2k+1)$ are positive integers, and each positive integer occurs exactly once as such an exponent. Hence the last series is equal to $\sum_{m=1}^{\infty} x^m = x/(1-x)$, and so $f(x) = x/(1-x)$. The value of the given sum is therefore $f(1/2) = 1$.

[An alternative solution (found by David Dueber, Kaushik Roy, and Ken Scheiwe) consists in showing, by induction, that the sum of the first n terms of the given series is $1 - (2^{2^n} - 1)^{-1}$. Since this expression tends to 1 as $n \to \infty$, the sum of the series is equal to 1.]

Problem 6

Let $f(n)$ denote the number of 1's in the binary expansion of n. Evaluate the sum

$$\sum_{n=1}^{\infty} \frac{f(n)}{n(n+1)}.$$

Solution. Let S denote the sum of the given series (which converges since $f(n) \le 1 + \log_2 n$). Using the relations $f(2n) = f(n)$ and $f(2n+1) = f(2n) + 1 = f(n) + 1$ and splitting the series into odd and even parts, we obtain

$$\begin{aligned} S &= \sum_{n=1}^{\infty} \frac{f(2n)}{(2n)(2n+1)} + \sum_{n=0}^{\infty} \frac{f(2n+1)}{(2n+1)(2n+2)} \\ &= \sum_{n=1}^{\infty} \frac{f(n)}{(2n)(2n+1)} + \sum_{n=0}^{\infty} \frac{f(n)+1}{(2n+1)(2n+2)} \\ &= \sum_{n=0}^{\infty} \left(\frac{1}{(2n+1)(2n+2)} \right) + \sum_{n=1}^{\infty} \left(\frac{f(n)}{(2n)(2n+1)} + \frac{f(n)}{(2n+1)(2n+2)} \right) \\ &= -\sum_{n=1}^{\infty} \frac{(-1)^n}{n} + \sum_{n=1}^{\infty} \frac{f(n)}{(2n)(n+1)} = \log 2 + \frac{1}{2} S. \end{aligned}$$

Hence, $S = \log 2 + S/2$, and so $S = 2 \log 2 = \log 4$.