UIUC UNDERGRADUATE MATH CONTEST

APRIL 15, 2000, 10 am – 1 pm

Problem 1

Suppose that a_1, a_2, $\ldots$, a_n are n given integers. Prove that there exist integers r and s with $0 \le r < s \le n$ such that $a_{r+1} + a_{r+2} + \ldots + a_s$ is divisible by n.

Problem 2

Let p be a point inside a triangle having sides of lengths a, b, c. Let h_a be the distance from p to the side of length a, and let h_b and h_c be defined analogously. Let $h := \min(h_a, h_b, h_c)$ and $s := (a+b+c)/2$. Prove that

$$h \le \sqrt{(s-a)(s-b)(s-c)/s}.$$

Problem 3

Let f and g be twice continuously differentiable functions on $[0, 1]$ with $f(0) = g(0) = 0 = f(1) = g(1)$. Suppose that $0 < f(x) < g(x)$ for $0 < x < 1$ and that $f''(x) < 0$ for $0 < x < 1$. Show that

$$\int_0^1 f'(x)^2\,dx \le \int_0^1 g'(x)^2\,dx.$$

Problem 4

Prove that if a, b, and c are odd positive integers, then the polynomial $ax^2 + bx + c$ has no rational roots.

Problem 5

Evaluate the infinite series

$$\frac{1}{2^1 - 2^{-1}} + \frac{1}{2^2 - 2^{-2}} + \frac{1}{2^4 - 2^{-4}} + \frac{1}{2^8 - 2^{-8}} + \cdots$$

Problem 6

Let $f(n)$ denote the number of 1's in the binary expansion of n. Evaluate the sum

$$\sum_{n=1}^{\infty} \frac{f(n)}{n(n+1)}.$$