The Fibonacci constant

Start with the famous Fibonacci sequence

\[1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, 1597, 2584, \ldots \]

(defined by \(F(1) = 1, F(2) = 1,\) and \(F(n) = F(n - 1) + F(n - 2)\) for \(n \geq 3\)). Drop all but the last digit in each of these numbers, and interpret the resulting sequence of digits as the decimal expansion of a certain real number \(F \in (0, 1)\), which we may term the “Fibonacci constant”:

\[F = 0.112358314594370774\ldots\]

In other words, \(F\) is the real number in the interval \((0, 1)\) whose \(n\)th digit after the decimal point is the last digit of the \(n\)-th Fibonacci number \(F(n)\). The question now is simple:

\[\text{Is } F \text{ rational?}\]

—Turn Page for Solution—
Solution to “The Fibonacci constant”

We will show that $F$ is rational. This is equivalent to showing that the decimal expansion of $F$ is eventually periodic.

By definition, $F = 0.f_1f_2f_3\ldots$, where $f_n$ is the last digit of the $n$-th Fibonacci number $F(n)$, i.e., $f_n$ is the residue of $F(n)$ modulo 10. In particular, we have $f_n \equiv F(n)$ modulo 10, and hence, by the recurrence for $F(n)$,

\[ f_{n+2} \equiv f_{n+1} + f_n \mod 10 \quad (n = 1, 2, \ldots). \tag{*} \]

Since $f_{n+2}$ must be an element of $\{0, 1, \ldots, 9\}$, the congruence \((*)\) determines $f_{n+2}$ uniquely, for any given pair of values $(f_n, f_{n+1})$. By induction it follows that any given pair of consecutive values $(f_n, f_{n+1})$ determines the entire remainder of the sequence, i.e., $f_{n+i}$ for all $i = 1, 2, \ldots$.

Now note that each pair $(f_n, f_{n+1})$ must be of the form $(a, b)$ with both $a, b \in \{0, 1, \ldots, 9\}$. Since there only finitely many (namely, $10^2$) such pairs $(a, b)$, by the pigeonhole principle there exist indices $m < n$ with $(f_m, f_{m+1}) = (f_n, f_{n+1})$. Setting $p = n - m$, we then have $f_{m+p+i} = f_{m+i}$ for $i = 0, 1$, and by the above observation, this relation persists for all $i = 0, 1, \ldots$. Thus the sequence $\{f_i\}$ is eventually periodic with period $p$, as we wanted to show.