Introduction to Analytic Number Theory

Math 531 Lecture Notes, Fall 2005

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Version 2013.01.07

Chapter 5

Distribution of primes II: Proof of the Prime Number Theorem

5.1 Introduction

In this chapter we give an analytic proof of the Prime Number Theorem (PNT) with error term. In its original form, the PNT is the assertion that the number of primes, $\pi(x)$, satisfies

$$\pi(x) \sim \frac{x}{\log x} \quad (x \to \infty), \tag{5.1}$$

but, as we have shown in Chapter 3, the PNT is equivalent to any one of the relations

$$\pi(x) \sim \mathrm{Li}(x) \quad (x \to \infty), \tag{5.2}$$

$$\theta(x) \sim x \quad (x \to \infty), \tag{5.3}$$

$$\psi(x) \sim x \quad (x \to \infty), \tag{5.4}$$

where

$$\mathrm{Li}(x) = \int_2^x \frac{dt}{\log t},$$

and

$$\theta(x) = \sum_{p \le x} \log p, \quad \psi(x) = \sum_{n \le x} \Lambda(n).$$

We will prove the PNT in the form (5.4); more precisely, we will establish the following quantitative form (i.e., one with explicit error term) of this relation.

Theorem 5.1 (Prime number theorem with error term)**.** *We have*

$$\psi(x) = x + O\left(x \exp\left(-c(\log x)^{\alpha}\right)\right) \quad (x \geq 2), \tag{5.5}$$

where c is a positive constant and $\alpha = 1/10$.

To gauge the quality of the error term, we note that, on the one hand,

$$x \exp\left(-c(\log x)^{\alpha}\right) \ll_k \frac{x}{(\log x)^k},$$

for any fixed constant k, while, on the other hand,

$$x \exp\left(-c(\log x)^{\alpha}\right) \gg_{\epsilon} x^{1-\epsilon}$$

for any fixed $\epsilon > 0$. (These estimates hold regardless of the specific value of α, as long as $0 < \alpha < 1$.)

The PNT was proved independently, and essentially simultaneously, by Jacques Hadamard and Charles de la Vallée Poussin at the end of the 19th century. The proofs of Hadamard and de la Vallée Poussin both used an analytic approach that had its roots in the work of Riemann some 50 years earlier.

After the PNT had been proved, the main focus shifted to establishing the PNT with as good an error term as possible. This problem is still wide open, and what we know is very far from what is being conjectured.

Tables 5.1 and 5.2 list the principal milestones in this development, and a more detailed description is given below. We will state all results in terms of the form (5.4) of the PNT, but the error terms in the relations (5.3) and (5.2) are essentially the same as for (5.4). (This is not true for the original form (5.1) of the PNT, since the right-hand side, $x/\log x$, is only a crude approximation to $\pi(x)$ that differs from the "true" size of $\pi(x)$ by a term of order $x/(\log x)^2$; the "correct" approximation for $\pi(x)$ is the logarithmic integral $\mathrm{Li}(x)$.)

Author(s)	**Bound for** $\psi(x) - x$	**Zerofree region** ($t^* = \max(\|t\|, 3)$)	**Remarks**
Chebyshev (1851)	$cx \quad (x \geq x_0)$ $(c \approx 0.1)$		Chebyshev bound
Hadamard, de la Vallée Poussin (1896)	$o(x)$	$\sigma \geq 1$	Prime Number Theorem
De la Vallée Poussin (1899)	$O\left(xe^{-c\sqrt{\log x}}\right)$	$\sigma \geq 1 - \frac{c}{\log t^*}$	"Classical" error term
Littlewood (1922)	$O\left(xe^{-c\sqrt{\log x \log\log x}}\right)$	$\sigma \geq 1 - \frac{c \log\log t^*}{\log t^*}$	
Vinogradov–Korobov (1958)	$O\left(xe^{-c\frac{(\log x)^{3/5}}{(\log\log x)^{1/5}}}\right)$	$\sigma \geq 1 - \frac{c(\log\log t^*)^{-1/3}}{(\log t^*)^{2/3}}$	Current record
	$O_\epsilon(x^{1/2+\epsilon})$, $\epsilon > 0$	$\sigma > 1/2$	Riemann Hypothesis

Table 5.1: The error term in the Prime Number Theorem, I

Author(s)	**Bound for** $\psi(x) - x$	**Remarks**
Erdös–Selberg (1949)	$o(x)$	First elementary proof
Bombieri, Wirsing (1964)	$O_A\left(x(\log x)^{-A}\right)$, any $A > 0$	First elementary proof with error term
Diamond–Steinig (1970)	$O_\alpha\left(xe^{-c(\log x)^\alpha}\right)$, any $\alpha < 1/7$	First elementary proof with exponential error term
Lavrik–Sobirov (1973)	$O_\alpha\left(xe^{-c(\log x)^\alpha}\right)$, any $\alpha < 1/6$	Current confirmed record for error term in elementary proof
	$O\left(xe^{-c\sqrt{\log x}}\right)$	Likely limit of elementary proofs

Table 5.2: The error term in the Prime Number Theorem, II: Elementary proofs

- **The classical error term.** (5.5) with $\alpha = 1/2$ was established by de la Vallée Poussin shortly after his proof of the PNT. This is a stronger result than the one we will prove here (with $\alpha = 1/10$), but to obtain this error term requires considerably more machinery from complex analysis than we have time to develop (such as the theory of entire functions of finite order, Hadamard products, and the theory of the Gamma function). The proof we will give goes back to E. Landau in the early part of the 20th century and has the advantage that it is a relatively "low tech" proof, requiring only a modest amount of complex analysis.

- **Vinogradov's error term.** The only significant improvement over de la Vallée Poussin's error term is due to I.M. Vinogradov who, some 50 years ago, obtained (5.5) with $\alpha = 3/5 - \epsilon$, for any fixed $\epsilon > 0$ (with the constant c depending on ϵ). Aside from minor improvements, in which the "ϵ" was made precise, Vinogradov's result still represents the current record in the error term of the PNT.

- **Error terms obtained by elementary methods.** The first "elementary" proof of the PNT was given by Erdős and Selberg in the 1940s. (Here "elementary" is to be interpreted in a technical sense—an elementary proof is one that avoids the use of tools from complex analysis. "Elementary" in this context is not synonymous with "simple"; in fact, the restriction to "elementary" methods comes at the expense of rendering the proof much longer, more complicated, and less transparent.)

  Other elementary proofs have since been given, but the early elementary proofs did not give explicit error terms, and most elementary approaches to the PNT yield only very weak error terms. It wasn't until the 1970s when Diamond and Steinig obtained a form of the PNT by elementary methods that involved an exponential error term as in (5.5), though only with an exponent $\alpha = 1/7 - \epsilon$, which is smaller than the exponents $\alpha = 1/2$ and $\alpha = 3/5 - \epsilon$ in the results of de la Vallée Poussin and Vinogradov. The current record for the value of α in elementary error terms is only slightly larger, namely $\alpha = 1/6 - \epsilon$. This still falls far short of the "classical" exponent $\alpha = 1/2$. Obtaining the value $\alpha = 1/2$ by elementary means would be a major achievement, and there are reasons to believe that this value represents the limit of what can possibly be achieved by elementary methods.

- **The conjectured error term.** Assuming primes behave, in some appropriate sense, randomly, one might expect the error term in (5.5) to be of size about the square root of the main term. Thus, a natural conjecture would be that $\psi(x) = x + O_\epsilon(x^{1/2+\epsilon})$ for every fixed $\epsilon > 0$. As we will see, this conjecture is equivalent to the Riemann Hypothesis. Moreover, if true, it is best-possible; i.e., the exponent $1/2$ here cannot be replaced by a smaller exponent. An indication of how far we are from proving such a result is the fact, noted above, that the error term in (5.5) is greater than $x^{1-\epsilon}$, for any fixed $\epsilon > 0$.

The proof of Theorem 5.1 will take up most of the remainder of this chapter. We now give a brief outline of the argument.

Our starting point is Perron's formula in the form given by Theorem 4.16 for the function $f(n) = \Lambda(n)$. Since $\Lambda(n)$ has Dirichlet series $-\zeta'(s)/\zeta(s)$, this formula gives

$$\psi_1(f,x) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\left(-\frac{\zeta'(s)}{\zeta(s)}\right)\frac{x^{s+1}}{s(s+1)}ds$$

for any $a > 1$, where

$$\psi_1(x) = \int_0^x \psi(y)dy.$$

We apply this formula initially with a value of a depending on x and slightly larger than 1 (namely, $a = 1 + 1/\log x$), and then move part of the line of integration to the left of the line $\sigma = 1$. Since $\zeta(s)$ has a pole at $s = 1$, the integrand has a pole at the same point, and passing over this pole we pick up a contribution $x^2/2$ from the residue of the integrand at $s = 1$. This contribution will be the main term in the estimate for $\psi_1(x)$. The error term will come from bounding the integral over the shifted path of integration. In order to obtain good estimates for the integrand, we need to, on the one hand, move as far to the left of $\sigma = 1$ as possible (so that $|x^s| = x^\sigma$ is small compared to x). On the other hand, since any zero of $\zeta(s)$ gives rise to a pole of $\zeta'(s)/\zeta(s)$, we can only move within a region that we know to be "zero-free", and in which we have good upper bounds for $|\zeta'(s)|$ and $1/|\zeta(s)|$. The region in which we can establish such bounds consists of points s bounded on the left by a curve of the form $\sigma = 1 - c(\log t)^{-9}$, which approaches the line $\sigma = 1$ asymptotically as $|t| \to \infty$. (Of course, if we knew RH, and had corresponding bounds for $|\zeta'(s)|$ and $1/|\zeta(s)|$, we could work in the larger region $\sigma > 1/2$.)

The establishment of a zero-free region and the proof of appropriate bounds for $1/\zeta$ and ζ' within this region will take up the bulk of the proof of Theorem 5.1. Once we have such bounds, the estimation of the complex integral is relatively easy, leading to a formula of the form $\psi_1(x) = x^2/2 + R(x)$ with an error term that is essentially (except for an extra factor x and a different value of the constant) that in (5.5) with $\alpha = 1/10$. An additional argument is then needed to translate this estimate into a similar one for $\psi(x)$.

Notation and conventions. Many of our estimates will involve constants. We will label these constants consecutively by $c_1, c_2, \ldots$ or $A_1, A_2, \ldots$.

Unless otherwise indicated, all constants are positive and absolute; i.e., they do not depending on any parameters and could, in principle, be given numerical values.

5.2 The Riemann zeta function, I: basic properties

Recall that for $\sigma > 1$ the Riemann zeta function is defined by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \quad (\sigma > 1),$$

i.e., $\zeta(s)$ is the Dirichlet series for the arithmetic function 1. We begin by collecting some elementary properties of this function, most of which have been established earlier.

Theorem 5.2 (Basic properties of the zeta function).

(i) *$\zeta(s)$ is analytic in $\sigma > 1$ and there has the Dirichlet series representation $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$.*

(ii) *$\zeta(s)$ has an analytic continuation to a function defined on the half-plane $\sigma > 0$ and analytic in this half-plane with the exception of a simple pole at $s = 1$ with residue 1. The analytic continuation is also denoted by $\zeta(s)$ and has the integral representation*

$$\zeta(s) = \frac{s}{s-1} - s\int_1^{\infty} \{x\} x^{-s-1} dx \quad (\sigma > 0). \tag{5.6}$$

(iii) *$\zeta(s)$ has an Euler product representation $\zeta(s) = \prod_p (1 - p^{-s})^{-1}$ in $\sigma > 1$.*

(iv) *$\zeta(s)$ has no zeros in the half-plane $\sigma > 1$.*

Proof. (i), (ii), and (iii) were established in Theorems 4.11 and 4.3; (iv) follows immediately from the Euler product representation, since the Euler product is absolutely convergent, and none of its factors is zero, in the half-plane $\sigma > 1$. (In general, an absolutely convergent infinite product can only be 0 if one of its factors is 0.) □

5.3 The Riemann zeta function, II: upper bounds

In this section we establish upper bounds for $\zeta(s)$ and $\zeta'(s)$ in a region that extends slightly to the left of the line $\sigma = 1$ into the critical strip. Recall that c_i and A_i denote positive constants.

Theorem 5.3 (Upper bounds for $\zeta(s)$ and $\zeta'(s)$)**.**

(i) $$|\zeta(s)| \leq 4\frac{|t|^{1-\sigma_0}}{1-\sigma_0} \quad (|t| \geq 2,\ 1/2 \leq \sigma_0 < 1, \sigma \geq \sigma_0)$$

(ii) $$|\zeta(s)| \leq A_1 \log |t| \quad (|t| \geq 2,\ \sigma \geq 1 - \frac{1}{4\log|t|})$$

(iii) $$|\zeta'(s)| \leq A_2 \log^2 |t| \quad (|t| \geq 2,\ \sigma \geq 1 - \frac{1}{12\log|t|})$$

To prove this result, we need three lemmas.

Lemma 5.4.

$$\zeta(s) = \sum_{n=1}^{N} \frac{1}{n^s} - \frac{N^{1-s}}{1-s} - s\int_N^\infty \{u\}u^{-s-1}du \quad (N \in \mathbb{N}, \sigma > 0). \tag{5.7}$$

Proof. The argument is modification of that used to establish the integral representation (5.6). Given a positive integer N, we apply the Mellin transform representation (Theorem 4.10) to the Dirichlet series

$$F(s) = \sum_{n=N+1}^{\infty} \frac{1}{n^s} = \zeta(s) - \sum_{n=1}^{N}\frac{1}{n^s}.$$

This is the Dirichlet series corresponding to the function f defined by $f(n) = 1$ if $n > N$ and $f(n) = 0$ if $n \leq N$, whose partial sums are given by $M(f,x) = [x] - N$ if $x \geq N$ and $M(f,x) = 0$ otherwise. Hence Theorem 4.10 gives, for $\sigma > 1$,

$$\begin{aligned} F(s) &= s\int_1^\infty M(f,x)x^{-s-1}dx = s\int_N^\infty ([x]-N)x^{-s-1}dx \\ &= s\int_N^\infty x^{-s}dx - sN\int_N^\infty x^{-s-1}dx - s\int_N^\infty \{x\}x^{-s-1}dx \\ &= -\frac{sN^{1-s}}{1-s} - N^{1-s} - s\int_N^\infty \{x\}x^{-s-1}dx \\ &= -\frac{N^{1-s}}{1-s} - s\int_N^\infty \{x\}x^{-s-1}dx. \end{aligned}$$

Since $\zeta(s) = F(s) + \sum_{n=1}^{N} n^{-s}$, this yields the desired relation (5.7) in the range $\sigma > 1$. Since both sides of this relation are analytic in $\sigma > 0$ except for a simple pole at $s = 1$ with residue 1 (note that the integral $\int_N^\infty \{x\}x^{-s-1}dx$ is uniformly convergent, and hence analytic, in any half-plane $\sigma \ge \epsilon$, $\epsilon > 0$), the relation remains valid in the larger half-plane $\sigma > 0$, as asserted. □

Lemma 5.5.

$$|\zeta(s)| \le \sum_{n=1}^{N} \frac{1}{n^\sigma} + \frac{N^{1-\sigma}}{|t|} + \frac{|s|}{\sigma} N^{-\sigma} \quad (N \in \mathbb{N}, \sigma > 0, t \ne 0). \tag{5.8}$$

Proof. This follows immediately from the previous lemma, on noting that each of the three terms on the right-hand side of (5.7) is bounded, in absolute value, by the corresponding term on the right of (5.8). For the first two terms, this is obvious, and for the third term this follows from the inequality

$$\left| s \int_N^\infty \{u\} u^{-s-1} du \right| \le |s| \int_N^\infty u^{-\sigma-1} du = \frac{|s| N^{-\sigma}}{|\sigma|}. \qquad \square$$

Lemma 5.6.

$$|\zeta(s)| \le \frac{N^{1-\sigma_0}}{1-\sigma_0} + \frac{N^{1-\sigma_0}}{|t|} + \left(1 + \frac{|t|}{\sigma_0}\right) N^{-\sigma_0} \tag{5.9}$$

$$(N \in \mathbb{N}, 1/2 < \sigma_0 < 1, \sigma \ge \sigma_0 > 0, t \ne 0).$$

Proof. We show that the three terms on the right of (5.8) are bounded by the corresponding terms in (5.9). Using the hypotheses $\sigma \ge \sigma_0$ and $\sigma_0 < 1$ and the inequality $n^{-\sigma} \le \int_{n-1}^{n} x^{-\sigma} dx$, we see that the first term on the right of (5.8) is at most

$$\sum_{n=1}^{N} \frac{1}{n^{\sigma_0}} \le 1 + \int_1^N x^{-\sigma_0} dx = 1 + \frac{N^{1-\sigma_0} - 1}{1-\sigma_0} \le \frac{N^{1-\sigma_0}}{1-\sigma_0},$$

as desired. Since $\sigma \ge \sigma_0$, the second term is trivially bounded by the corresponding term in (5.9). The same holds for the third term, in view of the bound $|s|/\sigma \le (\sigma + |t|)/\sigma \le 1 + |t|/\sigma_0$. Hence (5.9) follows from (5.8). □

Proof of Theorem 5.3. (i) We apply Lemma 5.6 with $N = [|t|]$, where $[x]$ denotes the greatest integer $\le x$. Since, by hypothesis, $0 < \sigma_0 < 1$, we then have $N^{1-\sigma_0} \le |t|^{1-\sigma_0}$, so the lemma gives

$$|\zeta(s)| \le \frac{|t|^{1-\sigma_0}}{1-\sigma_0} \left(1 + \frac{1-\sigma_0}{|t|} + \frac{1-\sigma_0}{[|t|]} + \frac{(1-\sigma_0)|t|}{\sigma_0 [|t|]} \right). \tag{5.10}$$

Since $|t| \geq 2$ we have $(1-\sigma_0)/|t| \leq (1-\sigma_0)/[|t|] \leq 1/2$. Moreover, the inequalities $[|t|] \geq |t|/2$ and $1/2 \leq \sigma_0 < 1$ give $(1-\sigma_0)|t|/(\sigma_0[|t|]) \leq 2$. Hence the expression in parentheses on the right of (5.10) is at most $1 + 1/2 + 1/2 + 2 = 4$, and we obtain (i).

(ii) We set $\sigma_0 = 1 - 1/(4\log|t|)$. Since $|t| \geq 2$, we have $4\log|t| \geq \log 16 > 2$ and so $1/2 < \sigma_0 < 1$. Hence we can apply the estimate of part (i) with this value of σ_0 and obtain

$$|\zeta(s)| \leq 4\frac{|t|^{1-\sigma_0}}{1-\sigma_0} = \frac{4e^{1/4}}{1/(4\log|t|)} = 16e^{1/4}\log|t|,$$

which is the desired bound with constant $A_1 = 16e^{1/4}$.

(iii) First note that, for $\sigma \geq 2$, the Dirichlet series representation of $\zeta'(s)$ implies $|\zeta'(s)| \leq \sum_{n=1}^{\infty}(\log n)n^{-2}$, so the asserted bound holds trivially in the half-plane $\sigma \geq 2$. Also, the analyticity of $\zeta(s)$ in the region $\{s : \operatorname{Re} s > 0, s \neq 1\}$ implies that $|\zeta'(s)|$ is uniformly bounded in any compact rectangle contained in this region. Hence the asserted bound also holds in the range $2 \leq |t| \leq 3$, $\sigma \geq 1/2$. It therefore remains to show that this bound holds (with a suitable choice of the constant A_2) when $|t| \geq 3$.

Let now s be given in the range $\sigma \geq 1 - 1/(12\log|t|)$, $|t| \geq 3$, and set $\delta = 1/(12\log|t|)$. Note that, since $|t| \geq 3 \geq e$, we have $\sigma > 1 - 1/12$, and $0 < \delta < 1/12$, so the disk $\{s' \in \mathbb{C} : |s'-s| \leq \delta\}$ is contained in the region of analyticity of $\zeta(s)$. We can therefore express $\zeta'(s)$ by Cauchy's theorem to get

$$|\zeta'(s)| = \left|\frac{1}{2\pi i}\oint_{|s'-s|=\delta}\frac{\zeta(s')}{(s'-s)^2}ds\right| \leq \frac{1}{\delta}\max_{|s'-s|=\delta}|\zeta(s')|. \tag{5.11}$$

To estimate the right-hand side of (5.11), we will show that for $|s'-s| \leq \delta$, $\zeta(s')$ is bounded by a constant multiple of $\log|t|$. We will do so by verifying that all s' in this range fall into the range of validity of the upper bound for $\zeta(s')$ established in part (ii).

Let $s' = \sigma' + it'$ with $|s'-s| \leq \delta$ be given. By our hypotheses $|t| \geq 3$ and $\sigma \geq 1-\delta$ we have

$$|t'| \geq |t| - \delta \geq |t| - 1/12 > 2$$

and

$$|t'| \leq |t| + \delta \leq |t| + 1/12 \leq \frac{13}{12}|t| \leq |t|^{3/2}.$$

Hence

$$\sigma' \geq \sigma - \delta \geq 1 - \frac{1}{6\log|t|} \geq 1 - \frac{1}{6\log|t'|^{2/3}} = 1 - \frac{1}{4\log|t'|}.$$

Thus the point s' lies in the range in which the bound (ii) is valid, and we therefore obtain

$$|\zeta(s')| \le A_1 \log|t'| \le (3/2)A_1 \log|t| \quad (|s'-s| = \delta).$$

Substituting this estimate in (5.11), we obtain

$$|\zeta'(s)| \le \frac{1}{\delta}(3/2)A_1 \log|t| = 18A_1(\log|t|)^2,$$

which is the desired estimate. □

5.4 The Riemann zeta function, III: lower bounds and zerofree region

The next result gives a zero-free region for $\zeta(s)$ to the left of the line $\sigma = 1$ of "width" a constant multiple of $(\log|t|)^{-9}$, and a lower bound for $\zeta(s)$ in this region. This result is the most important ingredient in the proof of the PNT; the value $\alpha = 1/10$ in the estimate (5.5) is directly related to the exponent 9 appearing in the definition of the region. De la Vallée Poussin's error term ($\alpha = 1/2$) is a consequence of a similar estimate, but in a wider region, with the exponent 1 instead of 9, and Vinogradov's value $\alpha = 3/5-\epsilon$ corresponds to an exponent $2/3+\epsilon$ in the zero-free region.

Theorem 5.7 (Zero-free region and upper bound for $1/\zeta(s)$)**.**

(i) *$\zeta(s)$ has no zeros in the closed half-plane $\sigma \ge 1$.*

(ii) *There exist constants $c_1 > 0$ and $A_3 > 0$ such that $\zeta(s)$ has no zeros in the region*

$$\sigma > 1 - c_1, \quad |t| \le 2,$$

and in this region satisfies

$$\left|\frac{1}{\zeta(s)}\right| \le A_3.$$

(iii) *There exist constants $c_2 > 0$ and $A_4 > 0$ such that $\zeta(s)$ has no zeros in the region*

$$\sigma \ge 1 - \frac{c_2}{(\log|t|)^9}, \quad |t| \ge 2,$$

and in this region satisfies

$$\left|\frac{1}{\zeta(s)}\right| \le A_4(\log|t|)^7.$$

The key ingredient in the proof is the following elementary inequality.

Lemma 5.8 (3-4-1 Lemma). *For any real number θ we have*

$$3 + 4\cos\theta + \cos(2\theta) \ge 0.$$

Proof. We have

$$\begin{aligned} 0 \le (1+\cos\theta)^2 &= 1 + 2\cos\theta + \cos^2\theta = 1 + 2\cos\theta + (1/2)(1+\cos(2\theta)) \\ &= (1/2)(3 + 4\cos\theta + \cos(2\theta)). \end{aligned}$$

□

We use this lemma to deduce a lower bound for a certain product of powers of the zeta function.

Lemma 5.9 (3-4-1 inequality for $\zeta(s)$). *We have*

$$\left|\zeta(\sigma)^3\zeta(\sigma+it)^4\zeta(\sigma+2it)\right| \ge 1 \quad (\sigma > 1, t \in \mathbb{R}).$$

Proof. Note that for $\operatorname{Re} s > 1$,

$$\begin{aligned} \log|\zeta(s)| &= \log\left|\prod_p \left(1 - \frac{1}{p^s}\right)^{-1}\right| = -\operatorname{Re}\sum_p \log\left(1 - \frac{1}{p^s}\right) \\ &= \operatorname{Re}\sum_p \sum_{m\ge 1} \frac{1}{mp^{ms}} = \sum_p \sum_{m\ge 1} \frac{\cos(t\log p^m)}{mp^{m\sigma}}, \end{aligned}$$

where, as usual, $\sigma = \operatorname{Re} s$ and $t = \operatorname{Im} s$, and log denotes the principal branch of the logarithm. Applying this relation with σ, $\sigma + it$, and $\sigma + 2it$ in place of s, we obtain

$$\begin{aligned} &\log\left|\zeta(\sigma)^3\zeta(\sigma+it)^4\zeta(\sigma+2it)\right| \\ &\quad = 3\log|\zeta(\sigma)| + 4\log|\zeta(\sigma+it)| + \log|\zeta(\sigma+2it)| \\ &\quad = \sum_p \sum_{m\ge 1} \frac{P(t\log p^m)}{mp^{m\sigma}}, \end{aligned}$$

where

$$P(\theta) = 3 + 4\cos\theta + \cos(2\theta)$$

is the trigonometric polynomial of Lemma 5.8. Since, by that lemma, $P(\theta)$ is nonnegative, all terms in the double series on the right are nonnegative, so the left-hand side is nonnegative as well. This implies the asserted inequality.

□

Proof of Theorem 5.7. (i) By Theorem 5.2 $\zeta(s)$ has no zeros in the *open* half-plane $\sigma > 1$, so it remains to exclude the possibility of a zero on the line $\sigma = 1$. We argue by contradiction and suppose that $\zeta(1+it_0) = 0$ for some real number t_0. Recall that, by Theorem 5.2, $\zeta(s)$ is analytic in the half-plane $\sigma > 0$, except for a simple pole at $s = 1$. Since ζ has a pole at $s = 1$, it cannot have a zero there, so we necessarily have $t_0 \neq 0$.

With a view towards applying Lemma 5.9, we consider the behavior of the three functions $\zeta(\sigma)$, $\zeta(\sigma + it_0)$, and $\zeta(\sigma + 2it_0)$ as $\sigma \to 1+$. Since $\zeta(s)$ has a pole at 1, $\zeta(\sigma)(\sigma - 1)$ is bounded as $\sigma \to 1+$. Furthermore, our assumption that $\zeta(1 + it_0) = 0$ implies, by the analyticity of $\zeta(s)$, that the expression

$$\frac{\zeta(\sigma + it_0)}{\sigma - 1} = \frac{\zeta(\sigma + it_0) - \zeta(1 + it_0)}{(\sigma + it_0) - (1 + it_0)}$$

also stays bounded as $\sigma \to 1+$. Finally, the analyticity of $\zeta(s)$ at $1 + 2it_0$ implies that $\zeta(\sigma+2it_0)$ converges to $\zeta(1+2it_0)$ as $\sigma \to 1+$, and, in particular, stays bounded. It follows that the function

$$\begin{aligned} &\left|\zeta(\sigma)^3\zeta(\sigma + it_0)^4\zeta(\sigma + 2it_0)\right| \\ &\quad = (\sigma - 1)\ \ |\zeta(\sigma)(\sigma - 1)|^3 \cdot \left|\frac{\zeta(\sigma + it_0)}{\sigma - 1}\right|^4 \cdot |\zeta(\sigma + 2it_0)| \end{aligned}$$

is of order $O(\sigma - 1)$ as $\sigma \to 1+$ and hence tends to 0. On the other hand, by Lemma 5.9, this function is bounded from below by 1, so we have arrived at a contradiction. Hence $\zeta(s)$ cannot have a zero on the line $\sigma = 1$.

(ii) First note that in the half-plane $\sigma \geq 2$ the asserted bound holds trivially: indeed, in this half-plane we have

$$|\zeta(s) \geq 1 - \sum_{n \geq 2} \frac{1}{n^2} = 1 - \left(\frac{\pi^2}{6} - 1\right) > 0$$

and thus

$$\left|\frac{1}{\zeta(s)}\right| \leq (2 - \pi^2/6)^{-1} \quad (\sigma \geq 2). \tag{5.12}$$

It remains therefore to show that $1/\zeta(s)$ is uniformly bounded in the compact rectangle

$$1 - c_1 \leq \sigma \leq 2, \quad |t| \leq 2, \tag{5.13}$$

with a sufficiently small positive constant c_1.

Now, by part (i), $\zeta(s)$ has no zeros in the closed half-plane $\sigma \geq 1$, so $1/\zeta(s)$ is analytic in this half-plane and therefore bounded in any compact region contained in this half-plane. In particular, $1/\zeta(s)$ is bounded in the rectangle $1 \leq \sigma \leq 2$, $|t| \leq 2$. By compactness, it follows that $1/\zeta(s)$ remains bounded in any sufficiently small neighborhood of this rectangle, and, in particular, in a rectangle of the form (5.13).

(iii) For $\sigma \geq 2$ the bound follows from (5.12), so we may restrict to the case when $\sigma \leq 2$. To obtain the desired bound for $1/\zeta(s)$ we will use again Lemma 5.9, in conjunction with the upper bounds for $\zeta(s)$ and $\zeta'(s)$ established in Theorem 5.3.

We fix a constant A that will be chosen later and let t be given with $|t| \geq 2$. We consider first the range

$$1 + A(\log|t|)^{-9} \leq \sigma \leq 2. \tag{5.14}$$

By Lemma 5.9 we have, for $\sigma > 1$,

$$|\zeta(\sigma + it)| \geq \frac{1}{\zeta(\sigma)^{3/4}} \cdot \frac{1}{|\zeta(\sigma + 2it)|^{1/4}}. \tag{5.15}$$

Since $\zeta(s)$ has a simple pole at $s = 1$, there exists an absolute constant c_3 such that

$$\zeta(\sigma) \leq c_3(\sigma - 1)^{-1}$$

for $1 < \sigma \leq 2$. Moreover, by Theorem 5.3(ii) we have

$$|\zeta(\sigma + 2it)| \leq A_1 \log|2t| \leq 2A_1 \log|t|,$$

where in the last step we have used the trivial inequality $\log(2|t|) \leq \log|t|^2 = 2\log|t|$, which is valid since $|t| \geq 2$. Inserting these bounds into (5.15) and now restricting to the narrower range (5.14), we obtain

$$\begin{aligned} |\zeta(\sigma + it)| &\geq c_3^{-3/4}(2A_1)^{-1/4}(\sigma - 1)^{3/4}(\log|t|)^{-1/4} \\ &\geq c_4 A^{3/4}(\log|t|)^{-7}, \end{aligned} \tag{5.16}$$

where $c_4 = c_3^{-3/4}(2A_1)^{-1/4}$ is an absolute constant.

This proves the asserted bound in the range (5.14), for any choice of the constant A. To complete the proof, we show that, if A is chosen sufficiently small, then a bound of the same type holds in the range

$$1 - A(\log|t|)^{-9} \leq \sigma \leq 1 + A(\log|t|)^{-9}. \tag{5.17}$$

Write

$$\sigma_1 = \sigma_1(A,t) = 1 - A(\log|t|)^{-9},$$
$$\sigma_2 = \sigma_2(A,t) = 1 + A(\log|t|)^{-9},$$

and note that for $\sigma_1 \le \sigma \le \sigma_2$ we have

$$\begin{aligned}|\zeta(\sigma+it)| &= \left|\zeta(\sigma_2+it) - \int_\sigma^{\sigma_2} \zeta'(u+it)du\right| \\ &\ge |\zeta(\sigma_2+it)| - (\sigma_2-\sigma_1) \max_{\sigma_1\le u\le\sigma_2} |\zeta'(u+it)|.\end{aligned}$$

Since $\sigma_2 + it$ falls in the range (5.14), the first term on the right can be estimated by (5.16). Moreover, by Theorem 5.3, we have $|\zeta'(s)| \le A_2(\log|t|)^2$ provided σ satisfies $\sigma \ge 1 - (12\log|t|)^{-1}$. If the constant A is sufficiently small, the range (5.17) is contained in the latter range, and so the above bound for $|\zeta'(s)|$ is valid in this range. We therefore obtain

$$\begin{aligned}|\zeta(\sigma+it)| &\ge c_4A^{3/4}(\log|t|)^{-7} - 2A(\log|t|)^{-9}A_2(\log|t|)^2 \\ &= A^{3/4}(c_4 - 2A^{1/4}A_2)(\log|t|)^{-7}.\end{aligned}$$

Choosing now A to be a small enough absolute constant, the coefficient of $(\log|t|)^{-7}$ in this bound becomes positive, and we obtain $|\zeta(\sigma+it)| \ge c_5(\log|t|)^{-7}$, with an absolute positive constant c_5. This gives the estimate asserted in (iii) with $c_2 = A$ and $A_4 = 1/c_5$. □

For later use, we record an easy consequence of the estimates of Theorems 5.7 and 5.3.

Theorem 5.10 (Upper bounds for $\zeta'(s)/\zeta(s)$). *There exist absolute positive constants* $0 < c_6 < 1/2$ *and* A_5 *such that for all* s *in the range*

$$\sigma \ge 1 - \frac{c_6}{(\log|t|)^9}, \quad |t| \ge 2, \tag{5.18}$$

we have

$$\left|\frac{\zeta'(s)}{\zeta(s)}\right| \le A_5(\log|t|)^9,$$

and for all s *in the range*

$$\sigma \ge 1 - c_6, \quad |t| \le 2, \quad s \ne 1, \tag{5.19}$$

we have

$$\left|\frac{\zeta'(s)}{\zeta(s)}\right| \le A_5 \max\left(1, \frac{1}{|\sigma-1|}\right).$$

Proof. The first estimate follows by combining the bounds (iii) of Theorems 5.7 and 5.3, and noting that the ranges of validity of these latter estimates, namely $\sigma \geq 1 - c_2(\log|t|)^{-9}$ and $\sigma \geq 1 - 1/(12\log|t|)$, both contain the range (5.18), provided $|t| \geq 2$ and the constant c_6 is chosen sufficiently small.

The second estimate is a consequence of the analytic properties of $\zeta(s)$: Since $1/\zeta(s)$ is analytic in $\sigma \geq 1$ and $\zeta(s)$ is analytic in $\sigma > 0$ except for a simple pole at $s = 1$, the logarithmic derivative $\zeta'(s)/\zeta(s)$ is analytic in $\sigma \geq 1$, except for a simple pole at $s = 1$. Hence $(s-1)\zeta'(s)/\zeta(s)$ is analytic in $\sigma \geq 1$. By compactness the analyticity extends to a region of the form $\sigma \geq 1 - c_6$, $|t| \leq 2$, provided c_6 is a sufficiently small constant. It follows that this function is bounded in the compact region $1 - c_6 \leq \sigma \leq 2$, $|t| \leq 2$, so that we have $|\zeta'(s)/\zeta(s)| \ll 1/|s-1| \leq 1/|\sigma - 1|$ in this region. Since for $\sigma \geq 2$, $\zeta'(s)/\zeta(s) = -\sum_{n\geq 1}\Lambda(n)n^{-s}$ is trivially bounded by $\sum_{n\geq 1}\Lambda(n)n^{-2} < \infty$, we obtain the second estimate of the theorem, by adjusting the constant A_5 if necessary. □

5.5 Proof of the Prime Number Theorem

We are now ready to prove the prime number theorem in the form given by Theorem 5.1. We break the proof into several steps:

Application of Perron's formula. We let

$$\psi_1(x) = \int_0^x \psi(y)dy = \sum_{n\leq x}\Lambda(n)(x-n),$$

and apply Perron's formula in the version given by Theorem 4.16 with $f(n) = \Lambda(n)$. The corresponding Dirichlet series is $F(s) = \sum_{n=1}^{\infty} f(n)n^{-s} = -\zeta'(s)/\zeta(s)$, which converges absolutely in $\sigma > 1$. Hence Perron's formula gives, for any $a > 1$ and any $x \geq 2$,

$$\psi_1(x) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\left(-\frac{\zeta'(s)}{\zeta(s)}\right)\frac{x^{s+1}}{s(s+1)}ds. \tag{5.20}$$

We fix $x \geq e$, let $e \leq T \leq x$ be a parameter that will be chosen later (as a function of x), and we set

$$a = 1 + \frac{1}{\log x}, \quad b = 1 - \frac{c_6}{(\log T)^9},$$

where c_6 is the constant of Theorem 5.10. Note that, since $c_6 < 1/2$ and $e \le T \le x$, we have

$$1 < a \le 2, \quad \frac{1}{2} < 1 - c_6 < b < 1. \tag{5.21}$$

Shifting the path of integration. The path of integration in (5.20) is a vertical line located within the half-plane $\sigma > 1$. We move the portion $|t| \le T$ of this path to the left of the line $\sigma = 1$, replacing it by a rectangular path joining the points $b \pm iT$ and $a \pm iT$. Thus, the new path of integration is of the form $L = \bigcup_{i=1}^{5} L_i$, with

$$\begin{aligned} L_1 &= (a - i\infty, a - iT], \\ L_2 &= [a - iT, b - iT], \\ L_3 &= [b - iT, b + iT], \\ L_4 &= [b + iT, a + iT], \\ L_5 &= [a + iT, a + i\infty). \end{aligned}$$

With this change of the path of integration we have

$$\psi_1(x) = M + \frac{1}{2\pi i} \sum_{j=1}^{5} I_j, \tag{5.22}$$

where M, the main term, is the contribution of the residues at singularities of the integrand in the region enclosed by the two paths and I_j denotes the integral over the path L_j.

The main term. The region enclosed by the original and the modified paths of integration is the rectangle with vertices

$$a \pm iT = 1 + \frac{1}{\log x} \pm iT, \quad b \pm iT = 1 - \frac{c_6}{(\log T)^9} \pm iT,$$

which falls within the zero-free region of $\zeta(s)$ given by Theorem 5.7. Thus, the integrand function has only one singularity in this region, namely that generated by the pole of $\zeta(s)$ at $s = 1$. Since this pole is simple, it follows that $-\zeta'(s)/\zeta(s)$ has a simple pole with residue 1 at $s = 1$, so the residue of the integrand function at this point is

$$\operatorname{Res}\left(-\frac{\zeta'(s)}{\zeta(s)} \cdot \frac{x^{s+1}}{s(s+1)}, s = 1\right) = \frac{1}{2}x^2.$$

Hence we have

$$M = \frac{1}{2}x^2. \tag{5.23}$$

This will be the main term of our estimate for $\psi_1(x)$. It remains to estimate the contribution of the integrals I_j. Here and in the remainder of this section, the constants implied in the $\ll$ notation are absolute and, in particular, independent of the value of T (which will only be chosen at the end of the proof).

Estimates of I_1 and I_5. These are the integrals along the vertical segments $(a - i\infty, a - iT]$ and $[a + iT, a + i\infty)$. On these segments we have $\sigma = a = 1 + 1/\log x$ and $|t| \geq T$. Thus,

$$\left|\frac{\zeta'(s)}{\zeta(s)}\right| \leq \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^a} = -\frac{\zeta'(a)}{\zeta(a)} \ll \frac{1}{a-1} = \log x,$$

and

$$\left|\frac{x^{s+1}}{s(s+1)}\right| = \frac{x^{a+1}}{|s||s+1|} \leq \frac{x^{a+1}}{t^2} = \frac{ex^2}{t^2}.$$

Hence we obtain the bounds

$$I_{1,5} \ll \int_T^{\infty} (\log x) \frac{x^2}{t^2} dt \ll \frac{x^2 \log x}{T}. \tag{5.24}$$

Estimates of I_2 and I_4. These are the integrals along the horizontal segments $[a - iT, b - iT]$ and $[b + iT, a + iT]$. By Theorem 5.10 we have on these paths

$$\left|\frac{\zeta'(s)}{\zeta(s)}\right| \ll (\log T)^9$$

and

$$\left|\frac{x^{s+1}}{s(s+1)}\right| \leq \frac{x^{a+1}}{|s||s+1|} \ll \frac{x^2}{T^2}.$$

Hence

$$I_{2,4} \ll \int_a^b (\log T)^9 \frac{x^2}{T^2} d\sigma \ll \frac{x^2 (\log T)^9}{T^2}. \tag{5.25}$$

Estimate of I_3. The remaining integral I_3 is the integral over the vertical segment $[b-iT, b+iT]$. By Theorem 5.10 and our choice $b = 1-c_6(\log T)^{-9}$, we have on this segment

$$\left|\frac{\zeta'(s)}{\zeta(s)}\right| \ll \max\left((\log T)^9, (1-b)^{-1}\right) \ll (\log T)^9.$$

Since

$$\left|\frac{x^{s+1}}{s(s+1)}\right| = \frac{x^{b+1}}{|s||s+1|} \ll x^{b+1}\min(1, t^{-2}),$$

we obtain the bound

$$I_3 \ll \int_{-T}^{T} x^{b+1}(\log T)^9 \min(1, t^{-2})dt \ll x^{b+1}(\log T)^9. \tag{5.26}$$

Estimation of $\psi_1(x)$. Substituting the estimates (5.23)–(5.26) into (5.22), we obtain

$$\psi_1(x) = \frac{1}{2}x^2 + R(x,T)$$

with

$$\begin{aligned} R(x,T) &\ll \sum_{j=1}^{5}|I_j| \ll x^2\left(\frac{\log x}{T} + \frac{(\log T)^9}{T^2} + x^{b-1}(\log T)^9\right) \\ &\ll x^2\left(\frac{\log x}{T} + (\log T)^9 \exp\left(-c_6\frac{\log x}{(\log T)^9}\right)\right), \end{aligned}$$

where in the last step we used the assumption $T \le x$, which implies that the term $(\log T)^9 T^{-2}$ is of smaller order than the term $(\log x)T^{-1}$ and hence can be dropped. We now choose T as

$$T = \exp\left((\log x)^{1/10}\right).$$

Since $x \ge e$, this choice satisfies our initial requirement on T, namely $e \le T \le x$, and we obtain

$$\begin{aligned} R(x,T) &\ll x^2\left((\log x)\exp\left(-(\log x)^{1/10}\right) + (\log x)^{9/10}\exp\left(-c_6(\log x)^{1/10}\right)\right) \\ &\ll x^2\exp\left(-c_7(\log x)^{1/10}\right), \end{aligned}$$

with a suitable positive constant c_7. (In fact, any constant less than c_6 will do.) Hence we have

$$\psi_1(x) = \frac{1}{2}x^2 + O\left(x^2\exp\left(-c_7(\log x)^{1/10}\right)\right) \quad (x \ge e). \tag{5.27}$$

Transition to $\psi(x)$. As the final step in the proof of the prime number theorem, we need to derive an estimate for $\psi(x)$ from the above estimate for $\psi_1(x)$. Recall that the two functions are related by $\psi_1(x) = \int_0^{\psi}(y)dy$. While from an estimate for a function one can easily derive a corresponding estimate for the integral of this function, a similar derivation in the other direction is in general not possible. However, in this case we are able to do so by exploiting the fact that the function $\psi(x) = \sum_{n\le x}\Lambda(n)$ is nondecreasing.

We fix $x \ge 6$ and a number $0 < \delta < 1/2$ (to be chosen later as a suitable function of x) and note that, by the monotonicity of $\psi(x)$, we have

$$\psi_1(x) - \psi_1(x(1-\delta)) = \int_{x(1-\delta)}^{x} \psi(y)dy \le \delta x\psi(x).$$

Since $x \ge x(1-\delta) \ge x/2 \ge 3 \ge e$ by our assumptions $x \ge 6$ and $\delta < 1/2$, we can apply (5.27) to each of the two terms on the left and obtain

$$\begin{aligned}\delta x\psi(x) &\ge \frac{1}{2}x^2 + O(x^2\Delta) - \frac{1}{2}x^2(1-\delta)^2 + O(x^2\Delta') \\ &= \delta x + O(x^2(\delta^2 + \Delta + \Delta')),\end{aligned} \tag{5.28}$$

where

$$\Delta = \exp\left(-c_7(\log x)^{1/10}\right), \quad \Delta' = \exp\left(-c_7(\log x(1-\delta))^{1/10}\right)$$

denote the *relative* error terms in (5.27), applied to x and $x' = x(1-\delta)$, respectively. Since $x(1-\delta) \ge x/2 \ge \sqrt{x}$, we have

$$(\log x(1-\delta))^{1/10} \ge (\log\sqrt{x})^{1/10} \ge (1/2)(\log x)^{1/10},$$

and hence $\Delta' \le \Delta^{1/2} \le 1$. With this inequality, (5.28) yields

$$\psi(x) \le x + O(x(\delta + \sqrt{\Delta}/\delta)).$$

Defining now δ by

$$\delta = \min(1/2, \Delta^{1/4}),$$

we obtain

$$\psi(x) \ge x + O(x\delta) = x + O\left(\exp\left(-c_8(\log x)^{1/10}\right)\right) \tag{5.29}$$

with $c_8 = c_7/4$.

A similar, but slightly simpler, argument starting from the inequality

$$\psi_1(x(1+\delta)) - \psi_1(x) \ge \delta x\psi(x).$$

shows that (5.29) also holds with the inequality sign reversed. Thus we have obtained the estimate

$$\psi(x) = x + O\left(\exp\left(-c_8(\log x)^{1/10}\right)\right)$$

in the range $x \geq 6$. Since the same estimate holds trivially for $2 \leq x \leq 6$, this completes the proof of the prime number theorem in the form (5.5) stated at the beginning of this section.

5.6 Consequences and remarks

Consequences of the PNT with error term. Using partial summation one can easily derive from the prime number theorem with the error term established here estimates for other prime number sums with comparable error terms. We collect the most important of these estimates in the following theorem.

Theorem 5.11 (Consequences of the PNT). *For $x \geq 2$ we have*

(i) $$\theta(x) = x + O(xR(x)),$$

(ii) $$\pi(x) = \mathrm{Li}(x) + O(xR(x)),$$

(iii) $$\sum_{p \leq x} \frac{\log p}{p} = \log x + C_1 + O(R(x)),$$

(iv) $$\sum_{p \leq x} \frac{1}{p} = \log\log x + C_2 + O(R(x)),$$

(v) $$\prod_{p \leq x} \left(1 - \frac{1}{p}\right) = \frac{e^{-\gamma}}{\log x}\left(1 + O(R(x))\right),$$

where $\mathrm{Li}(x) = \int_2^x \frac{dt}{\log t}$, the C_i are absolute constants, and $R(x)$ is an error term of the same type as in Theorem 5.1, except possibly for the value of the constant in the exponent, i.e., $R(x) = \exp(-c(\log x)^{1/10})$, with c a positive constant (not necessarily the same as in Theorem 5.1).

Proof. Estimate (i) follows from the $\psi(x)$ version of the PNT (i.e., Theorem 5.1) and the estimate

$$\begin{aligned} 0 \leq \psi(x) - \theta(x) &= \sum_{p \leq \sqrt{x}} \sum_{m=2}^{[(\log x)/(\log p)]} \log p \\ &\leq \sum_{p \leq \sqrt{x}} \log p \frac{\log x}{\log p} = \pi(\sqrt{x}) \log x \ll \sqrt{x} \ll xR(x). \end{aligned}$$

Estimates (ii)–(iv) can be deduced from (i) by a routine application of partial summation. We omit the details, and only note that the process typically results in a small loss in the constant in the exponent in $R(x)$. This is because one has to apply the PNT with values $y \le x$ in place of x and use estimates such as

$$\exp\{-c(\log y)^\alpha\} \le \exp\left\{-c2^{-\alpha}(\log x)^\alpha\right\} \quad (\sqrt{x} \le y \le x),$$

to bound error terms at y in terms of error terms at x.

The estimate (v) is a sharper version of Mertens' formula. Except for the value of the constant, this estimate follows from (iv), on noting that

$$\begin{aligned} -\sum_{p\le x} \log\left(1-\frac{1}{p}\right) - \sum_{p\le x}\frac{1}{p} &= \sum_{p\le x}\sum_{m=2}^{\infty}\frac{1}{mp^m} \\ &= \sum_{p}\sum_{m=2}^{\infty}\frac{1}{mp^m} + O\left(\sum_{p>x}\frac{1}{p^2}\right) = C + O\left(\frac{1}{x}\right), \end{aligned}$$

where C is a constant. That the constant on the right of (v) must be equal to $e^{-\gamma}$ follows from Mertens' formula (Theorem 3.4(iv)). □

It is important to note that an analogous sharpening does not hold for the original form $\pi(x) \sim x/\log x$ of the prime number theorem. In fact, combining the estimate (ii) of Theorem 5.11 with the asymptotic estimate for the logarithmic integral $\mathrm{Li}(x)$ given in Theorem 2.1 shows that $\pi(x)$ differs from $x/\log x$ by a term that is asymptotic to $x/\log^2 x$; more precisely, we have:

Corollary 5.12. *For any fixed positive integer k we have*

$$\pi(x) = \sum_{i=1}^{k}\frac{(i-1)!x}{(\log x)^i} + O_k\left(\frac{x}{(\log x)^{k+1}}\right) \quad (x \ge 2). \tag{5.30}$$

Estimates for the Moebius function. As we have seen in Chapter 3, the PNT in its asymptotic form $\psi(x) \sim x$ is equivalent to the asymptotic relation $M(\mu, x) = \sum_{n\le x}\mu(n) = o(x)$. It is reasonable to expect that a sharper form of the PNT would translate to a corresponding sharpening of the estimate for $M(\mu, x)$. This is indeed the case, and we have:

Theorem 5.13 (Moebius sum estimate)**.** *For $x \ge 2$, we have*

$$\sum_{n\le x}\mu(n) \ll xR(x),$$

where $R(x)$ is defined as in Theorem 5.11.

This result can be proved by essentially repeating the proof of the last section, with the function $-\zeta'(s)/\zeta(s)$, the Dirichlet series for $\Lambda(n)$, replaced by $1/\zeta(s)$, the Dirichlet series for $\mu(n)$. The argument goes through without problems with this modification, and, indeed, is simpler in some respects. The main difference is that the function $1/\zeta(s)$, unlike $\zeta'(s)/\zeta(s)$, is analytic at $s = 1$, so no main term appears when estimating the corresponding Perron integral. We omit the details.

Zero-free regions of the zeta function and the error term in the prime number theorem. The proof of the prime number theorem given here depended crucially on the existence of a zero-free region for the zeta function and bounds for $\zeta(s)$ and $1/\zeta(s)$ within this region. It is easy to see that a larger zero-free region, along with corresponding zeta bounds in this region, would lead to a better error term. In turns out that, in some sense, the converse also holds: a smaller error term in the prime number theorem implies the existence of a larger zero-free region for the zeta function. Indeed, there are results that go in both directions and give equivalences between zero-free regions and error terms. We state, without proof, one simple result of this type and derive several consequences from it.

Theorem 5.14. *Let $0 < \theta < 1$. Then the following are equivalent:*

(i) The Riemann zeta function has no zeros in the half-plane $\sigma > \theta$.

(ii) The prime number theorem holds in the form

$$\psi(x) = x + O_\epsilon(x^{\theta+\epsilon}) \quad (x \geq 2)$$

for every fixed $\epsilon > 0$.

The Riemann Hypothesis is the statement that $\zeta(s)$ has no zeros in the half-plane $\sigma > 1/2$. Taking $\theta = 1/2$ in the above result, we therefore obtain the following equivalence to the Riemann Hypothesis.

Corollary 5.15. *The Riemann Hypothesis holds if and only if, for every $\epsilon > 0$,*

$$\psi(x) = x + O_\epsilon(x^{1/2+\epsilon}) \quad (x \geq 2).$$

Since it is known that the Riemann zeta function has infinitely many zeros on the line $\sigma = 1/2$, condition (i) in Theorem 5.14 cannot hold with $\theta < 1/2$. By the equivalence of (i) and (ii) it follows that the same is true for condition (ii); that is, we have:

Corollary 5.16. *Given any $\epsilon > 0$, the estimate*

$$\psi(x) = x + O(x^{1/2-\epsilon}) \quad (x \geq 2)$$

does not hold.

5.7 Further results

The results on the Riemann zeta function and the PNT that we have proved in this chapter represent only a small part of what is known in this connection. The Riemann zeta function is one of the most thoroughly studied "special" functions in mathematics, and entire books have been devoted to this function. Even though the most famous problem about this function, the Riemann Hypothesis, remains open, there exists a well-developed theory of the Riemann zeta function, and its connection to the PNT. In this section, we present, without proof, some of the major known results, as well as some of the main conjectures in this area.

The functional equation and analytic continuation. A fundamental property of the zeta function that is key to any deeper study of this function is the functional equation it satisfies:

$$\zeta(1-s) = 2(2\pi)^{-s}\Gamma(s)\cos(\pi s/2)\zeta(s). \tag{5.31}$$

Here $\Gamma(s)$ is the so-called Gamma function, a meromorphic function that interpolates factorials in the sense that $\Gamma(n) = (n-1)!$ when n is a positive integer. The Gamma function is analytic in the half-plane $\sigma > 0$ and there has integral representation $\Gamma(s) = \int_0^\infty e^{-x}x^{s-1}dx$.

The functional equation relates values of $\zeta(s)$ to values $\zeta(1-s)$. The vertical line $\sigma = 1/2$ acts as an axis of symmetry for this functional equation, in that if s lies in the half-plane to the right of this line, then $1-s$ falls in the half-plane to the left of this line.

By Theorem 4.11, $\zeta(s)$ has an analytic continuation to the half-plane $\sigma > 0$. Similar arguments could be used to obtain a continuation to $\sigma > -1$, and, by induction, to $\sigma > -n$, for any positive integer n. However, the functional equation (5.31) provides an analytic continuation to the entire complex plane in a single step. To see this, note that function on the right of the equation is analytic in $\sigma > 0$ (the pole of $\zeta(s)$ at $s = 1$ is cancelled out by a zero of $\cos(\pi s/2)$ at the same point). Hence the same must be true for the function on the left. But this means that $\zeta(1-s)$ has an analytic continuation to the half-plane $\sigma > 0$, or, equivalently, that $\zeta(s)$ has an analytic continuation to the half-plane $\sigma < 1$.

Analytic properties of the Riemann zeta function. The key analytic properties (both known and conjectured) of the Riemann zeta function (henceforth considered as a meromorphic function on the entire complex plane), are the following:

- **Poles:** $\zeta(s)$ has a single pole at $s = 1$, with residue 1, and is analytic elsewhere.
- **Trivial zeros:** $\zeta(s)$ has simple zeros at the points $s = -2n$, $n = 1, 2, \dots$. These are called **trivial zeros**, as they are completely understood and have no bearing on the distribution of primes.
- **Nontrivial zeros:** All other zeros of $\zeta(s)$ are located in the **critical strip** $0 < \sigma < 1$. These zeros, commonly denoted by $\rho = \beta + i\gamma$, are closely related to the error term in the prime number theorem. They are symmetric with respect to both the **critical line** $\sigma = 1/2$, and the real axis. It is known that there are infinitely many nontrivial zeros, and good estimates are available for the number of such zeros up to a given height T, but their horizontal distribution within the strip $0 < \sigma < 1$ remains largely a mystery.
- **The Riemann Hypothesis:** The Riemann Hypothesis (RH) is the assertion that all nontrivial zeros lie exactly on the critical line $\sigma = 1/2$. This has been numerically verified for the first several billion nontrivial zeros (when ordered by increasing imaginary part). The closest theoretical approximation to the Riemann Hypothesis are zero-free regions of the form $\sigma > 1 - c(\log|t|)^{-\alpha}$, $|t| \geq 2$, for suitable exponents of α, the current record being Vinogradov's value of $\alpha = 2/3 + \epsilon$.
- **The Lindelöf Hypothesis:** Another well-known conjecture, though not quite as famous as RH, is the Lindelöf Hypothesis (LH), which states that the bound $\zeta(1/2 + it) \ll_\epsilon |t|^\epsilon$ holds for every fixed $\epsilon > 0$ and $|t| \geq 1$. It is easy to show that a bound of this type holds with exponent $1/2$; the current record for the exponent is approximately 0.16. It is known that LH follows from RH, though the proof of this implication is not easy.

Approximation of $\zeta(s)$ by partial sums. One of the more remarkable results on the zeta function is that, even though the series representation $\zeta(s) = \sum_{n=1}^\infty n^{-s}$ is only valid in $\sigma > 1$ (in fact, the series does not even converge when $\sigma \le 1$), an *approximate* version of this representation remains valid to the left of the line $\sigma = 1$. Here is a typical result of this nature:

Theorem 5.17 (Approximate formula for $\zeta(s)$). *For $1/2 \le \sigma \le 2$ and $|t| \ge 2$ we have*

$$\zeta(s) = \sum_{n \le |t|} \frac{1}{n^s} + O\left(|t|^{-\sigma}\right).$$

This estimate can be deduced from the identity (5.7). In fact, applying this identity with $N = [|t| + 1]$ gives, under the conditions $1/2 \le \sigma \le 2$ and $|t| \ge 2$ and assuming (without loss of generality) that $|t|$ is not an integer,

$$\left|\zeta(s) - \sum_{n \le |t|} \frac{1}{n^s}\right| \ll \frac{[|t|+1]^{1-\sigma}}{|1-s|} + |s| \left|\int_{[|t|+1]}^{\infty} \{u\} u^{-s-1} du\right|,$$

The first term on the right is of the desired order $\ll |t|^{-\sigma}$. Using the trivial bound $|\{u\}u^{-s-1}| \le u^{-\sigma-1}$ in the integral would give for the second term the bound $|s||t|^{-\sigma}/\sigma \ll |t|^{1-\sigma}$. This is too weak, but a more careful estimate of the integral, using integration by parts and the estimate $\int_0^x \{u\} du = (1/2)x + O(1)$, shows that this term is also of order $\ll |t|^{-\sigma}$.

Explicit formulae. The proof of the prime number theorem given in the last section clearly showed the significance of the zeros of the zeta function in obtaining sharp versions of the prime number theorem. However, the effect of possible zeros in this proof was rather indirect: A zero of the zeta function leads to a singularity in the Dirichlet $-\zeta'(s)/\zeta(s)$, thus creating an obstacle to moving the path of integration further to the left. Since the quality of the error term depends largely on how far to the left one can move the path of integration, the presence of zeros with real part close to 1 puts limits on the error terms one can obtain with this argument.

There are results, known as **explicit formulae**, that make the connection between prime number estimates much more explicit. We state two of these formulae in the following theorems.

Theorem 5.18 (Explicit formula for $\psi_1(x)$). *We have, for $x \ge 1$,*

$$\psi_1(x) = \frac{x^2}{2} - \sum_{\rho} \frac{x^{\rho+1}}{\rho(\rho+1)} - \frac{\zeta'(0)}{\zeta(0)} x + \frac{\zeta'(-1)}{\zeta(-1)} - \sum_{n=1}^{\infty} \frac{x^{1-2n}}{(2n)(2n-1)}, \tag{5.32}$$

where ρ runs through all nontrivial zeros of $\zeta(s)$.

Theorem 5.19 (Explicit formula for $\psi(x)$)**.** *We have, for $x \geq 2$ and any $2 \leq T \leq x$*

$$\psi(x) = x - \sum_{|\gamma| \leq T} \frac{x^\rho}{\rho} + O\left(\frac{x \log^2 x}{T}\right), \tag{5.33}$$

where $\rho = \beta + i\gamma$ runs through all nontrivial zeros of $\zeta(s)$ of height $|\gamma| \leq T$.

It is known that

$$\sum_\rho \frac{1}{|\rho|^2} < \infty, \tag{5.34}$$

while

$$\sum_\rho \frac{1}{|\rho|} = \infty, \tag{5.35}$$

so the series over ρ in the first explicit formula (for $\psi_1(x)$) converges absolutely, but not that in the second formula (for $\psi(x)$). Hence the need for working with a truncated version of this series in the latter case.

Explicit formulas can be used to translate results or conjectures on zeros of the zeta function to estimates for the prime counting functions $\psi(x)$ or $\psi_1(x)$. We illustrate this with two corollaries.

Corollary 5.20. *Assuming the Riemann Hypothesis, we have*

$$\psi_1(x) = \frac{x^2}{2} + O(x^{3/2}).$$

Proof. This follows immediately from (5.32) and (5.34) on noting that, under the Riemann Hypothesis, $|x^\rho| = x^{\mathrm{Re}\,\rho} = x^{1/2}$ for all nontrivial zeros ρ. □

The second corollary shows the effect of a single hypothetical zero that violates the Riemann Hypothesis. We first note that, due to the symmetry of the (nontrivial) zeros of the zeta function, zeros off the line $\sigma = 1/2$ come in quadruples: If $\rho = \beta + i\gamma$ is a zero of the zeta function with $1/2 < \beta \leq 1$, then so are $\beta - i\gamma$ and $1 - \beta \pm i\gamma$. Thus, it suffices consider zeros $\rho = \beta + i\gamma$ in the quadrant $\beta \geq 1/2$ and $\gamma \geq 0$.

Corollary 5.21. *Suppose that there exists exactly one zero $\rho = \beta + i\gamma$ of the zeta function with $1/2 < \beta \leq 1$, $\gamma \geq 0$, so that all zeros except $\beta \pm i\gamma$ and $1 - \beta \pm i\gamma$ have real part $1/2$. Then*

$$\psi_1(x) = \frac{x^2}{2} + cx^{1+\beta} \cos(\gamma \log x) + O(x^{3/2}),$$

where c is a constant depending on ρ.

Proof. As before, the contribution of the zeros satisfying the Riemann Hypothesis to the sum over ρ in (5.32) is of order $O(x^{3/2})$, and the same holds for the contribution of the zeros $1-\beta \pm i\gamma$ since $1-\beta < 1/2$. Thus, the only remaining terms in this sum are those corresponding to $\rho = \beta \pm i\gamma$. Their contribution is

$$\frac{x^{1+\beta+i\gamma}}{(\beta+i\gamma)(\beta+1+i\gamma)} + \frac{x^{1+\beta-i\gamma}}{(\beta-i\gamma)(\beta+1-i\gamma)},$$

and a simple calculation shows that this term is of the form $cx^{1+\beta}\cos(\gamma\log x)$. □

Thus, under the hypothesis of the corollary, $\psi_1(x)$ oscillates around the main term $x^2/2$ with an amplitude $cx^{1+\beta}$. In particular, a zero *on* the line $\sigma = 1$, i.e., with $\beta = 1$, would result in an oscillatory term of the form $cx^2\cos(\gamma\log x)$ in the estimate for $\psi_1(x)$, and thus contradict the PNT (though not Chebyshev's estimate, if the constant c is smaller than 1).

5.8 Exercises

5.1 Show that, if x is sufficiently large, then the interval $[2, x]$ contains more primes than the interval $(x, 2x]$.

5.2 Obtain an asymptotic estimate for the sum

$$S(x) = \sum_{x<p\leq 2x} \frac{1}{p}$$

with *relative* error $1/\log x$ (i.e., an estimate of the form $S(x) = f(x)(1 + O(1/\log x))$ with a simple elementary function $f(x)$.

5.3 Define $A(x)$ by $\pi(x) = x/(\log x - A(x))$. Show that $A(x) = 1 + O(1/\log x)$ for $x \geq 2$.

Remark. This result is of historical interest for the following reason: While the function $x/\log x$ is asymptotically equal to $\pi(x)$ by the prime number theorem, examination of numerical data suggests that the function $x/\log x$ is not a particularly good approximation to $\pi(x)$. Therefore, in the early (pre-PNT) history of prime number theory several other functions were suggested as suitable approximations to $\pi(x)$. In particular, Legendre proposed the function $x/(\log x - 1.08366)$ (The particular value of the constant 1.08366 was presumably obtained by some kind of regression analysis on the data.) On the other hand, Gauss suggested that $x/(\log x - 1)$ was a better match to $\pi(x)$. The problem settles this dispute, showing that Gauss had it right.

5.4 Let $f(n) = \Lambda(n) - 1$. Show that the Dirichlet series $F(s) = \sum_{n=1}^{\infty} f(n)n^{-s}$ converges for every s on the line $\sigma = 1$, and obtain an estimate for the rate of convergence, i.e., the difference $F(s) - \sum_{n\leq x} f(n)n^{-s}$, when $s = 1 + it$ for some fixed t. (The estimate may depend on t, but try to get as good an error term as possible assuming the PNT with exponential error term.)

5.5 Let $0 < \alpha < 1$ be fixed. Show that if

(1) $$\theta(x) = x + O\left(x \exp\{-c(\log x)^{\alpha}\}\right) \quad (x \geq 2)$$

with some positive constant c, then

(2) $$\pi(x) = \mathrm{Li}(x) + O\left(x \exp\{-c'(\log x)^{\alpha}\}\right) \quad (x \geq 2)$$

with some (other) positive constant c' (but the same value of the exponent α).

5.6 Let $M(x) = \sum_{n\le x} \mu(n)$. Using complex integration as in the proof of the PNT, show that $M(x) = O(x\exp(-c(\log x)^{\alpha}))$, where $\alpha = 1/10$ and c is a positive constant.

5.7 Evaluate the integral

$$I_k(y) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{y^s}{s^k}\,ds,$$

where k is an integer ≥ 2, and y and c are positive real numbers. Then use this evaluation to derive a Perron type formula for the integral

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F(s)\frac{x^s}{s^k}\,ds,$$

where $F(s) = \sum_{n=1}^{\infty} f(n)n^{-s}$ is a Dirichlet series, stating any conditions that are needed for this formula to be valid.

5.8 Let $F(s) = \sum_p p^{-s}$, where the summation runs over all primes. Show that $F(s) = \log\zeta(s) + G(s)$, where log denotes the principal branch of the logarithm, and $G(s)$ is analytic in the half-plane $\sigma > 1/2$. Deduce from this that the function $F(s)$ does *not* have a meromorphic continuation to the left of the line $\sigma = 1$.