Asymptotic Analysis

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i

Chapter 1

Introduction: What is asymptotic analysis?

Much of undergraduate mathematics is about *exact* formulas and identities. For example, in calculus we are taught how to evaluate (in terms of "elementary" functions) integrals of certain classes of functions, such as all rational functions, or powers of trigonometric functions; in discrete mathematics we learn to evaluate (in "closed form") certain sums (e.g., $\sum_{k=0}^{n} k^2$ or $\sum_{k=0}^{n} \binom{n}{k}^2$); in combinatorics we learn formulas for various combinatorial quantities, such as the total number of subsets of a set; in number theory we learn how to evaluate various number-theoretic functions such as the Euler phi function; and in differential equations we learn how to exactly solve certain types of differential equations.

In the real world, however, the situations where an exact solution or formula exists are the exception rather than the rule. In many problems in analysis, combinatorics, probability, number theory and other areas, one encounters quantities (functions) that arise naturally and are worth studying, but for which no (simple) exact formula is known, and where it is unreasonable to expect that such a formula exist.[1]

What one often can do in such cases is to derive so-called *asymptotic estimates* for these quantities, i.e., approximations of the quantities by "simple" functions, which show how these quantities behave "asymptotically" (i.e.,

[1] In some cases it can even be *proved* that no such formula exists. For example, there is a remarkable result ("Risch theorem") that shows that a certain class of functions cannot be integrated in terms of elementary functions. This class of functions includes the functions $(\log x)^{-1}$ and e^{-x^2}, whose integrals define the logarithmic integral and the error function integral mentioned in the next chapter.

when the argument tends to infinity). In applications such approximations are often just as useful as an exact formula would be.

Asymptotic analysis, or "asymptotics", is concerned with obtaining such asymptotic estimates; it provides an array of tools and techniques that can be helpful for that purpose. In some cases obtaining asymptotic estimates is straightforward, but there are many situations where this can be quite difficult and involve a fair amount of ad-hoc analysis.

We now describe some problems from different areas of mathematics that can be approached by methods of asymptotic analysis and which are representative of the types of problems that are amenable to asymptotic analysis. All of these problems we will consider in detail later in this course.

Combinatorics: Stirling's formula. Perhaps the most famous asymptotic estimate is "Stirling's formula" for factorials. In its most basic form it reads[2]

$$n! \sim \sqrt{2n\pi} n^n e^{-n}.$$

A more precise version is

$$n! = \sqrt{2n\pi} n^n e^{-n} \left(1 + O\left(\frac{1}{n}\right)\right).$$

This formula is of great importance in combinatorics (and related areas such as probability) since many combinatorial quantities can be expressed in terms of binomial coefficients, and thus, ultimately, in terms of factorials.

Why should care about such a formula? After all, the factorial function is given by a simple and completely explicit formula, $n! = 1 \cdot 2 \cdots n$. One reason is that the factorials grow so rapidly that even for values n of modest size, their computation by hand or on a pocket calculator is impractical. Stirling's formula allows one to get a pretty good estimate for these numbers using a simple back-of-the-envelope type calculation. For example, using a crude form of Stirling's formula (namely, $\log n! \sim n \log n$) one can easily see that $10^{10}!$ has approximately 10^{11} decimal digits.

At a more fundamental level, Stirling's formula allows one to resolve theoretical questions that hinge on the precise rate of growth of factorials. For example, the d-dimensional *drunkard's walk problem* in probability leads to the series $\sum_{n=1}^{\infty} a_n^d$, where $a_n = 2^{-2n}\binom{2n}{n}$, and whether or not this series converges determines the answer to this problem. Using Stirling's formula one can show that $a_n \sim 1/\sqrt{2\pi n}$; hence the series converges for dimensions

[2]For precise definitions of the terminology and notations $\sim$ and O employed here see the following chapter.

$d \geq 3$, but diverges for dimensions one and two. A consequence of this is that, for dimensions $d = 1$ and $d = 2$ a drunkard taking random unit steps along a d-dimensional grid is guaranteed (with probability one) to eventually return to his starting location if the dimension d is at most 2, while in dimensions 3 or greater there is a positive probability that the drunkard gets "lost in space".

Number theory: the n-th prime. Because of the erratic nature of the distribution of primes there exists no exact "closed" formula for p_n, the n-th prime. However, one can derive *approximate* formulas, or "asymptotic estimates". For example, we know that p_n "is asymptotic to" $n \log n$, i.e.,

$$p_n \sim n \log n,$$

and, somewhat more precisely that

$$p_n = n \log n + O(n).$$

Such "asymptotic estimates" are less precise than an exact formula, but they still provide useful information about the size of p_n; for instance, the above estimate implies that the series $\sum_{n=1}^{\infty} 1/p_n$ diverges, while the series $\sum_{n=1}^{\infty} 1/(p_n \log n)$ converges.

Combinatorics: Harmonic numbers. Harmonic numbers are the partial sums of the harmonic series, $H_n = \sum_{k=1}^{n} 1/k$. These numbers arise naturally in a variety of combinatorial and probabilistic problems.[3]

Despite their rather simple definition there is no "closed formula" for these numbers. Thus, the best one can do is to give estimates for H_n. In fact, a classical application of asymptotic analysis gives the estimate

$$H_n = \log n + \gamma + O\left(\frac{1}{n}\right),$$

where $\gamma = 0.5772\ldots$ is the Euler constant.

[3]One such problem is the "coupon problem": Given N different coupons and assuming that each cereal box contains a coupon chosen at random from these N coupons, what is the expected number (in the sense of probabilistic expectation) of cereal boxes you need to buy in order to obtain a complete set of coupons? The answer is NH_N.

Number theory/analysis: The logarithmic integral. The function $\mathrm{Li}(x) = \int_2^x (\log t)^{-1} dt$ is called the logarithmic integral. It arises in number theory as the "best" approximation to the prime counting function $\pi(x)$, the number of primes up to x. However, while $\mathrm{Li}(x)$, in contrast to the prime counting function, is a smooth function with nice analytic properties such as monotonicity and infinite differentiability, it itself is a bit of a mystery as it cannot be evaluated in terms of elementary functions. Nonetheless, the behavior of $\mathrm{Li}(x)$ can be described quite well by methods of asymptotic analysis: One has, for example, the simple estimate

$$\mathrm{Li}(x) = \frac{x}{\log x} + O\left(\frac{x}{(\log x)^2}\right).$$

In fact, $\mathrm{Li}(x)$ has a representation in the form of an "asymptotic series" (see the following chapter for the precise definition of this concept):

$$\mathrm{Li}(x) \sim x \sum_{n=1}^{\infty} \frac{(n-1)!}{(\log x)^n}.$$

Analysis/probability: The error function integral. The integral $E(x) = \int_x^\infty e^{-t^2} dt$ is of importance in probability and statistics, as it represents (up to a constant factor) the tail probability in a normal distribution. As in the case of $\mathrm{Li}(x)$, the integral representing $E(x)$ cannot be evaluated exactly, but methods of asymptotic analysis lead to quite accurate approximations to $E(x)$ in terms of elementary functions; for example,

$$E(x) = \frac{1}{2x} e^{-x^2} \left(1 + O\left(\frac{1}{x}\right)\right).$$

Analysis: The Gamma function and Stirling's formula. The Gamma function is defined as $\Gamma(x) = \int_0^\infty e^{-t} x^{t-1} dt$. It satisfies $\Gamma(n) = (n-1)!$ when n is a positive integer and in this sense interpolates the factorial function. It arises naturally in many contexts in number theory, probability, and analysis. The integral defining $\Gamma(x)$ cannot be evaluated exactly in terms of elementary functions, but using asymptotic techniques one can obtain good estimates for $\Gamma(x)$ as $x \to \infty$. Since $n! = \Gamma(n+1)$ for positive integers n, such estimates can be used to establish Stirling's formula.

Analysis: The W-function. This function is defined via the equation $We^W = x$, which arises in a variety of contexts. For example, the solution

x to $x^x = y$ is given by $x = e^{W(\log y)}$. Also, the infinite "tower" $x^{x^{x^{\cdot}}}$ is equal to $e^{-W(-\log x)}$ in the range it converges.

The equation $We^W = x$ cannot be solved "explicitly" for W, but it is easy to see that, for any positive real number x, the equation has a unique real solution W, and thus defines a function $W = W(x)$, which has become known as the W-function. While an exact formula for W does not exist, one can derive asymptotic estimates for W; for example,

$$W(x) = \log x - \log\log x + O\left(\frac{\log\log x}{\log x}\right).$$

Analysis: Behavior of Laplace and Fourier transforms. Given a function $f(t)$ defined on the positive real axis, the function $F(x) = \int_0^\infty f(t)e^{-xt}dt$ (assuming the integral converges) is called the Laplace transform of $f(t)$. Similarly, if f is defined on the real line, the Fourier transform of f is defined as $\hat{f}(x) = \int_{-\infty}^\infty f(t)e^{-itx}dt$ whenever the integral exists. While for some functions $f(t)$ the integrals representing these transforms can be evaluated exactly, in general there is no such "closed" formula, even if the function $f(t)$ has a very simple form. In any case, methods of asymptotic analysis can usually be successfully employed to obtain information on the behavior of such transforms (e.g., their rate of decay as $x \to \infty$ or their rate of growth as $x \to 0$).

Combinatorics/probability: binomial sums. Many combinatorial quantities and discrete probabilities can be expressed as sums over binomial coefficients. For example, the probability that in n tosses of a fair coin there are between a and b heads is given by $2^{-n}\sum_{k=a}^b \binom{n}{k}$. Such expressions can rarely be evaluated exactly, so instead one tries obtain good approximations to such sums using methods of asymptotic analysis. In the above example, such an analysis leads to the famous De Moivre-Laplace theorem in probability theory.

Number theory: The partition function. The partition function, $p(n)$, denotes the number of "partitions" of n, i.e., the number of ways to express n as a sum of positive integers, disregarding order. For example, $p(4) = 5$ since 4 has 5 such representations: $4 = 4$, $4 = 3 + 1$, $4 = 2 + 2$, $4 = 2 + 1 + 1$, $4 = 1 + 1 + 1 + 1$. In contrast to some related combinatorial quantities (e.g., the corresponding counting function when the order is taken into account), there is no simple formula for $p(n)$, and the behavior of $p(n)$ as $n \to \infty$ is a classical and quite difficult problem in number theory. This problem was solved some 100 years ago by G.H. Hardy and S. Ramanujan, who showed, in a remarkable tour de force, the asymptotic formula

$$p(n) \sim \frac{c_1}{n} e^{c_2 \sqrt{n}},$$

where $c_1 = 1/(4\sqrt{3})$ and $c_2 = \pi\sqrt{2/3}$ are constants.