Problem 1

Call a set of integers a *PC-set* if it has the property that any pair of distinct elements of the set is coprime. Given $x \geq 2$, let $N(x) = \max\{|A| : A \subset [2, x], A \text{ is a PC-set}\}$. In other words, $N(x)$ is the maximal number of integers with the PC property that one can fit in the interval $[2, x]$. Prove that $N(x)$ is equal to $\pi(x)$, the number of primes $\leq x$. 
Problem 2

Let

\[ I(x, \alpha) = \int_{1}^{x} \frac{\sin(\alpha t)}{t} dt, \]

where \( \alpha \) is a fixed real and non-zero number. Use integration by parts to show that \( I(x, \alpha) \) converges as \( x \to \infty \), with limit \( I(\alpha) \), say, and show that \( I(x, \alpha) = I(\alpha) + O(1/x) \).
Problem 3

Let $f(x)$ and $g(x)$ be positive, continuous functions on $[0, \infty)$, and set $F(x) = \int_0^x f(y)\,dy$, $G(x) = \int_0^x g(y)\,dy$.

(i) Show (by a counterexample) that the relation

$$f(x) = o(g(x)) \quad (x \to \infty)$$

does not imply

$$F(x) = o(G(x)) \quad (x \to \infty).$$

(ii) **Bonus question:** Find (with proof) an appropriate general condition on $g(x)$ under which the implication $f(x) = o(g(x)) \Rightarrow F(x) = o(G(x))$. 

**Remark:** It is trivial to show that, if “$o$” is replaced by “$O$” in (1) and (2), then the implication holds. In other words, one can “pull out” a $O$-sign from an integral (provided the integrand is positive).
Problem 4

Obtain an estimate for the sum $\sum_{n \leq x} \frac{\log n}{n}$ with error term $O((\log x)/x)$. 
Problem 5

Let $\sum_{n=1}^{\infty} a_n$ be a convergent series with nonnegative terms $a_n$.

(i) Show that there exists a real-valued function $\psi(n)$ with $\lim_{n \to \infty} \psi(n) = \infty$ such that the series $\sum_{n=1}^{\infty} a_n \psi(n)$ still converges.

(ii)* Show that the conclusion holds even without the assumption that the terms $a_n$ be nonnegative (i.e., assuming only that the series $\sum_{n=1}^{\infty} a_n$ converges). (This requires a different, and more complicated argument.)
Problem 6*

Show that if \( f(x) \) satisfies \( f(x) = x^2 + O(x) \), and \( f \) is differentiable with nondecreasing derivative \( f'(x) \) for sufficiently large \( x \), then \( f'(x) = 2x + O(\sqrt{x}) \).

Remark. While \( O \)-estimates can be integrated provided the range of integration is contained in the range of validity of the estimate, in general such estimates cannot be differentiated. The above problem illustrates a situation where, under certain additional conditions (namely, the monotonicity of the derivative), differentiation of a \( O \)-estimate is allowed.
Problem 7*

Let $n$ be an integer $\geq 2$ and $p$ a positive real number. In class it was shown that (in the case $n = 2$, but the same argument works for general $n$)

$$\left(\sum_{i=1}^{n} a_i\right)^p \asymp_{n,p} \sum_{i=1}^{n} a_i^p \quad (a_1, a_2, \ldots, a_n > 0).$$

By the definition of the notation $\asymp_{n,p}$, this means that there exist positive constants $c_1(n, p)$ and $c_2(n, p)$ such that

$$c_1(n, p) \sum_{i=1}^{n} a_i^p \leq \left(\sum_{i=1}^{n} a_i\right)^p \leq c_2(n, p) \sum_{i=1}^{n} a_i^p \quad (a_1, a_2, \ldots, a_n > 0).$$

Determine the best-possible values for these constants.