Variance, covariance, correlation, moment-generating functions

[In the Ross text, this is covered in Sections 7.4 and 7.7. See also the Chapter Summary on pp. 405–407.]

- **Variance:**
  - **Definition:** $\mathrm{Var}(X) = \mathrm{E}(X^2) - E(X)^2 (= E(X - E(X))^2)$
  - **Properties:** $\mathrm{Var}(c) = 0$, $\mathrm{Var}(cX) = c^2 \mathrm{Var}(X)$, $\mathrm{Var}(X + c) = \mathrm{Var}(X)$
- **Covariance:**
  - **Definition:** $\mathrm{Cov}(X, Y) = E(XY) - E(X)E(Y) (= E(X - E(X))(Y - E(Y)))$
  - **Properties:**
    * **Symmetry:** $\mathrm{Cov}(X, Y) = \mathrm{Cov}(Y, X)$
    * **Relation to variance:** $\mathrm{Var}(X) = \mathrm{Cov}(X, X)$, $\mathrm{Var}(X+Y) = \mathrm{Var}(X)+\mathrm{Var}(Y)+2\,\mathrm{Cov}(X, Y)$
    * **Bilinearity:** $\mathrm{Cov}(cX, Y) = \mathrm{Cov}(X, cY) = c\,\mathrm{Cov}(X, Y)$,
      $\mathrm{Cov}(X_1 + X_2, Y) = \mathrm{Cov}(X_1, Y) + \mathrm{Cov}(X_2, Y)$,
      $\mathrm{Cov}(X, Y_1 + Y_2) = \mathrm{Cov}(X, Y_1) + \mathrm{Cov}(X, Y_2)$.
    * **Product formula:** $\mathrm{Cov}(\sum_{i=1}^n X_i, \sum_{j=1}^m Y_j) = \sum_{i=1}^n \sum_{y=1}^m \mathrm{Cov}(X_i, Y_j)$
- **Correlation:**
  - **Definition:** $\rho(X, Y) = \frac{\mathrm{Cov}(X,Y)}{\sqrt{\mathrm{Var}(X)\,\mathrm{Var}(Y)}}$
  - **Properties:** $-1 \le \rho(X, Y) \le 1$
- **Moment-generating function:**
  - **Definition:** $M(t) = M_X(t) = \mathrm{E}(e^{tX})$
  - **Computing moments via mgf's:** The derivates of $M(t)$, evaluated at $t = 0$, give the successive "moments" of a random variable X: $M(0) = 1$, $M'(0) = \mathrm{E}(X)$, $M''(0) = \mathrm{E}(X^2)$, $M'''(0) = \mathrm{E}(X^3)$, etc.
  - **Special cases:** (No need to memorize these formulas.)
    * X **standard normal:** $M(t) = \exp\{\frac{t^2}{2}\}$ (where $\exp(x) = e^x$)
    * X **normal** $N(\mu, \sigma^2)$**:** $M(t) = \exp\{\mu t + \frac{\sigma^2 t^2}{2}\}$
    * X **Poisson with parameter** λ**:** $M(t) = \exp\{\lambda(e^t - 1)\}$
    * X **exponential with parameter** λ**:** $M(t) = \frac{\lambda}{\lambda - t}$ for $|t| < \lambda$.
  - **Notes:** In contrast to expectation and variance, which are numerical constants associated with a random variable, a moment-generating function is a *function* in the usual (one-variable) sense (see the above examples). A moment generating function characterizes a distribution uniquely, and thus provides an additional way (in addition to the p.d.f. and c.d.f.) to describe a distribution.

Additional properties of independent random variables

If X and Y are independent, then the following additional properties hold:

- $E(XY) = E(X)E(Y)$. More generally, $E(f(X)g(Y)) = E(f(X))E(g(X))E(f(Y))$.
- $M_{X+Y}(t) = M_X(t)M_Y(t)$
- $\mathrm{Var}(X+Y) = \mathrm{Var}(X) + \mathrm{Var}(Y)$
- $\mathrm{Cov}(X,Y) = 0$, $\rho(X,Y) = 0$

Notes:

- Analogous properties hold for three or more random variables; e.g., if $X_1, \dots, X_n$ are *mutually independent*, then $E(X_1 \dots X_n) = E(X_1)\dots E(X_n)$, and $\mathrm{Var}(\sum_{i=1}^n X_i) = \sum_{i=1}^n \mathrm{Var}(X_i)$.
- Note that the product formula for mgf's involves the *sum* of two independent r.v.'s, not the product. The reason behind this is that the definition of the mgf of $X+Y$ is the expectation of $e^{t(X+Y)}$, which is equal to the product $e^{tX} \cdot e^{tY}$. In case of indepedence, the expectation of that product is the product of the expectations.
- While for independent r.v.'s, covariance and correlation are always 0, the converse is not true: One can construct r.v.'s X and Y that have 0 covariance/correlation 0 ("uncorrelated"), but which are not independent.