Independence

Definition and properties

1. **Independence of two events:** A and B are called independent if they satisfy the *product formula*
$$P(AB) = P(A)P(B).$$

2. **Independence of three or more events:** A, B, C are called **mutually independent** if the product formula holds for (i) the intersection of all three events (i.e., $P(ABC) = P(A)P(B)P(C)$) and (ii) for any combination of two of these three events (i.e., $P(AB) = P(A)P(B)$ and similarly for $P(AC)$, $P(BC)$). More generally, n events $A_1, \dots, A_n$ are called independent if the product formula holds for any subcollection of these events.

3. **Independence of complements:** If A and B are independent, then so are A and B^c, A^c and B, and A^c and B^c. More generally, if $A_1, \dots, A_n$ are independent, then so are the events $A_1', \dots, A_n'$, where each A_i' is either A_i or A_i^c.

4. **Connection between independence and conditional probability:** If the conditional probability $P(A|B)$ is equal to the ordinary ("unconditional") probability $P(A)$, then A and B are independent. Conversely, if A and B are independent, then $P(A|B) = P(A)$ (assuming $P(B) > 0$).

Notes and hints

- **Formal versus intuitive notion of independence:** When working problems, always use the above formal mathematical definitions of independence and conditional probabilities. While these definitions are motivated by our intuitive notion of these concepts and *most of the time* consistent with what our intuition would predict, intuition, aside from being non-precise, does fail us some time and lead to wrong conclusions as illustrated, for example, by the various paradoxes in probability.

- **Independence is not the same as disjointness:** If A and B are disjoint (corresponding to mutually exclusive events), then the intersection AB is the empty set, so $P(AB) = P(\emptyset) = 0$, so independence can only hold in the trivial case when one of the events has probability 0. While, at first glance, this might seem counterintuitive, it is, in fact, consistent with the interpretation of disjointness as meaning that A and B are mutually exclusive, that is, if A occurs, then B cannot occur, and vice versa.

- **Independence in Venn diagrams:** In contrast to other properties such as disjointness, independence can **not** be spotted in Venn diagrams.

- **Don't make assumptions about independence:** If a problem does not explicitly state that two events are independent, they are probably not, and not you should not make any assumptions about independence.