Name: Section (circle one): 9 am 10 am

Math 461, Sections B/C, Spring 2009
HW Assignment 10, due Friday, 4/24/2009

Instructions

- **Write your name on the cover sheet and staple the sheet to the assignment.** Do the problems in order, and make sure that each problem is clearly labelled. The assignment is due in class on the above due date.
- **About this assignment.** This assignment is on the class material through Monday, April 20. It covers variance, covariance, correlation, and moment-generating functions; see the handout from April 17, and Sections 7.4 and 7.7 of the Ross text. **There are a total of 10 problems, the last few on the back of the page.**

HW 10 Problems

1. Suppose $\mathrm{Var}(X) = 5,000$, $\mathrm{Var}(Y) = 10,000$, and $\mathrm{Var}(X + Y) = 17,000$.
   Find $\mathrm{Var}(X + 1.1Y + 100)$.
2. Suppose $E(X) = 5$, $E(X^2) = 27.4$, $E(Y) = 7$, $E(Y^2) = 51.4$ and $\mathrm{Var}(X + Y) = 8$.
   Find $\mathrm{Cov}(X + Y, X + 1.2Y)$.
3. (Problem #37 in Chapter 7) Let X denote the sum, and Y the difference of the numbers on two rolls of a die. Find $\mathrm{Cov}(X, Y)$. (Hint: This can be worked out directly, but it's quicker to rewrite X and Y as $X = X_1 + X_2$ and $Y = X_1 - X_2$, where X_i is the number on roll i, and then use the properties of covariance.)
4. (Problem #38 in Chapter 7) Suppose X and Y have joint density
   $$f(x, y) = \frac{2e^{-2x}}{x}, \quad 0 < x < \infty, 0 < y \leq x.$$
   Find $\mathrm{Cov}(X, Y)$.
5. Let $X_1, X_2, \ldots$ be mutually independent random variables, each having mean μ and variance σ^2, and let $S_n = \sum_{i=1}^{n} X_i$. Find $\mathrm{Cov}(S_k, S_n)$ for general integers k and n with $1 \leq k \leq n$.
6. (a) Let X be the number showing up on a single roll of a die. Find the mgf (moment-generating function) $M_X(t)$ of X. Simplify/evaluate sums! (Hint: Geometric series.)
   (b) Now suppose you roll a die **three** times, and let X be the sum of the numbers showing up on the three rolls. Find the mgf of X.
7. Suppose a discrete r.v. has mgf
   $$M(t) = \frac{2}{5}e^{t} + \frac{1}{5}e^{2t} + \frac{2}{5}e^{3t}.$$
   (a) Find the expectation and variance of X.
   (b) Find the distribution of X, i.e., the values x of X and their probabilities. (Hint: "Reverse-engineer" the construction of an mgf from a given distribution!)

8. Suppose X has density

$$f(x) = Ce^{-3x}, \quad 1 \le x < \infty,$$

with some constant C. (Note the range for x starts at $x = 1$.) Determine the constant C, and find the mgf $M_X(t)$ of X.
Hint: In this case, the integral giving the mgf does not necessarily converge. However, it converges when t is small enough, and this is all one usually needs (e.g., to compute derivatives of $M(t)$ at 0). Make sure to pay attention to the convergence of the integral involved, and state clearly why the integral converges, and under what conditions on t.

9. A company insures homes in three cities, City J, City K, and City L. The losses occurring in these cities, J, K, L, are independent. The moment-generating functions for the loss distributions of the cities are

$$M_J(t) = (1-2t)^{-3}, \quad M_K(t) = (1-2t)^{-2.5}, \quad M_L(t) = (1-2t)^{-4.5}$$

Let X represent the combined losses from the three cities, i.e., $X = J + K + L$. Calculate $E(X^3)$.

10. In Friday's class the problem of approximating a given r.v. Y by a linear function $a + bX$ was described, the solution was outlined, and the optimal choices of the coefficients a and b were given in terms of the correlation ρ of X and Y and the means and variances μ_X, μ_Y, σ_X^2, σ_Y^2. This problem furnishes the details of the analysis. With X, Y as above, let

$$f(a,b) = E((Y - (a + bX))^2).$$

As described in class, we use this function as a measure of the quality of the fit, and we seek to minimize it with respect to a and b.

(a) Expand $f(a,b)$, and express it in terms of the quantities μ_X, μ_Y, σ_X, σ_Y, and ρ, and the variables a and b.

(b) For fixed b find the unique a, say $\hat{a}$, that minimizes $f(a,b)$. Express $\hat{a}$ in terms of the above quantities μ_X, etc.

(c) Now set $a = \hat{a}$, and find the unique value of b, say $\hat{b}$, that minimizes $f(\hat{a}, b)$. Again express $\hat{b}$ in terms of the above quantities μ_X, etc.

(d) Find $f(\hat{a}, \hat{b})$, i.e., the value of $E(Y - (a + bX))^2)$ when a and b are chosen optimally, $a = \hat{a}$ and $b = \hat{b}$.