HW Assignment 10, due Friday, 4/24/2009

Instructions

• Write your name on the cover sheet and staple the sheet to the assignment. Do the problems in order, and make sure that each problem is clearly labelled. The assignment is due in class on the above due date.

• About this assignment. This assignment is on the class material through Monday, April 20. It covers variance, covariance, correlation, and moment-generating functions; see the handout from April 17, and Sections 7.4 and 7.7 of the Ross text. There are a total of 10 problems, the last few on the back of the page.

HW 10 Problems

1. Suppose \( \text{Var}(X) = 5,000 \), \( \text{Var}(Y) = 10,000 \), and \( \text{Var}(X + Y) = 17,000 \).
   Find \( \text{Var}(X + 1.1Y + 100) \).

2. Suppose \( E(X) = 5 \), \( E(X^2) = 27.4 \), \( E(Y) = 7 \), \( E(Y^2) = 51.4 \) and \( \text{Var}(X + Y) = 8 \).
   Find \( \text{Cov}(X + Y, X + 1.2Y) \).

3. (Problem #37 in Chapter 7) Let \( X \) denote the sum, and \( Y \) the difference of the numbers on two rolls of a die. Find \( \text{Cov}(X, Y) \). (Hint: This can be worked out directly, but it’s quicker to rewrite \( X \) and \( Y \) as \( X = X_1 + X_2 \) and \( Y = X_1 - X_2 \), where \( X_i \) is the number on roll \( i \), and then use the properties of covariance.)

4. (Problem #38 in Chapter 7) Suppose \( X \) and \( Y \) have joint density
   \[
   f(x, y) = \frac{2e^{-2x}}{x}, \quad 0 < x < \infty, 0 < y \leq x.
   \]
   Find \( \text{Cov}(X, Y) \).

5. Let \( X_1, X_2, \ldots \) be mutually independent random variables, each having mean \( \mu \) and variance \( \sigma^2 \), and let \( S_n = \sum_{i=1}^{n} X_i \). Find \( \text{Cov}(S_k, S_n) \) for general integers \( k \) and \( n \) with \( 1 \leq k \leq n \).

6. (a) Let \( X \) be the number showing up on a single roll of a die. Find the mgf (moment-generating function) \( M_X(t) \) of \( X \). Simplify/evaluate sums! (Hint: Geometric series.)
   (b) Now suppose you roll a die three times, and let \( X \) be the sum of the numbers showing up on the three rolls. Find the mgf of \( X \).

7. Suppose a discrete r.v. has mgf
   \[ M(t) = \frac{2}{5}e^t + \frac{1}{5}e^{2t} + \frac{2}{5}e^{3t}. \]
   (a) Find the expectation and variance of \( X \).
   (b) Find the distribution of \( X \), i.e., the values \( x \) of \( X \) and their probabilities. (Hint: “Reverse-engineer” the construction of an mgf from a given distribution!)
8. Suppose \( X \) has density
\[
f(x) = Ce^{-3x}, \quad 1 \leq x < \infty,
\]
with some constant \( C \). (Note the range for \( x \) starts at \( x = 1 \).) Determine the constant \( C \), and find the mgf \( M_X(t) \) of \( X \).

**Hint:** In this case, the integral giving the mgf does not necessarily converge. However, it converges when \( t \) is small enough, and this is all one usually needs (e.g., to compute derivatives of \( M(t) \) at 0). Make sure to pay attention to the convergence of the integral involved, and state clearly why the integral converges, and under what conditions on \( t \).

9. A company insures homes in three cities, City J, City K, and City L. The losses occurring in these cities, \( J, K, L \), are independent. The moment-generating functions for the loss distributions of the cities are
\[
M_J(t) = (1 - 2t)^{-3}, \quad M_K(t) = (1 - 2t)^{-2.5}, \quad M_L(t) = (1 - 2t)^{-4.5}
\]

Let \( X \) represent the combined losses from the three cities, i.e., \( X = J + K + L \). Calculate \( E(X^3) \).

10. In Friday’s class the problem of approximating a given r.v. \( Y \) by a linear function \( a + bX \) was described, the solution was outlined, and the optimal choices of the coefficients \( a \) and \( b \) were given in terms of the correlation \( \rho \) of \( X \) and \( Y \) and the means and variances \( \mu_X, \mu_Y, \sigma_X^2, \sigma_Y^2 \). This problem furnishes the details of the analysis. With \( X, Y \) as above, let
\[
f(a, b) = E((Y - (a + bX))^2).
\]

As described in class, we use this function as a measure of the quality of the fit, and we seek to minimize it with respect to \( a \) and \( b \).

(a) Expand \( f(a, b) \), and express it in terms of the quantities \( \mu_X, \mu_Y, \sigma_X, \sigma_Y \), and \( \rho \), and the variables \( a \) and \( b \).

(b) For fixed \( b \) find the unique \( a \), say \( \hat{a} \), that minimizes \( f(a, b) \). Express \( \hat{a} \) in terms of the above quantities \( \mu_X \), etc.

(c) Now set \( a = \hat{a} \), and find the unique value of \( b \), say \( \hat{b} \), that minimizes \( f(\hat{a}, b) \). Again express \( \hat{b} \) in terms of the above quantities \( \mu_X \), etc.

(d) Find \( f(\hat{a}, \hat{b}) \), i.e., the value of \( E(Y - (a + bX))^2 \) when \( a \) and \( b \) are chosen optimally, \( a = \hat{a} \) and \( b = \hat{b} \).