Math 461 C, Spring 2009
Final Exam Solutions and Comments

1. The following parts are independent of each other.

10 pts

(a) What is the coefficient of $x^9y^{10}z^{11}$ if $(x+y+z)^{30}$ is expanded? Express this coefficient in terms of powers and/or factorials (and not, for example, binomial coefficients).

Solution. By the multinomial theorem, this coefficient is equal to

$$\binom{30}{9,10,11} = \boxed{\frac{30!}{9!10!11!}}$$

10 pts

(b) Evaluate the sum

$$\sum_{k=0}^{30} \binom{30}{k} \frac{1}{2^k}.$$

Solution. By the binomial theorem,

$$\sum_{k=0}^{30} \binom{30}{k} \frac{1}{2^k} = \left(1 + \frac{1}{2}\right)^{30} = \boxed{\left(\frac{3}{2}\right)^{30}}$$

10 pts

(c) Given a standard poker deck (52 cards, 4 suits of 13 cards each), what is the probability that a standard 5-card poker hand consists of *two pairs* (i.e., two cards of one value (e.g., Ace), two cards of another value, and one card of a third value)?

Solution. There are $\binom{13}{2}$ ways to pick the two double values, $\binom{11}{1}$ ways to pick the single value, $\binom{4}{2}^2$ ways to pick suits for the two double values, and $\binom{4}{1}$ ways to pick the suit for the single value. Multiplying these counts and dividing by the total number of 5-card poker hands, $\binom{52}{5}$, gives the probability for two pairs:

$$P(\text{two pairs}) = \frac{1}{\binom{52}{5}} \binom{13}{2}\binom{11}{1}\binom{4}{2}^2\binom{4}{1}$$

2. Suppose $P(A) = 1/4$, $P(B) = 1/3$, and $A \subset B$.

(a) Find $P(A \mid B)$.

6 pts

Solution. Since $A \subset B$, we have $AB = A$, so

$$P(A \mid B) = \frac{P(AB)}{P(B)} = \frac{P(A)}{P(B)} = \frac{1/4}{1/3} = \boxed{\frac{3}{4}}$$

(b) Find $P(B \mid A)$.

6 pts

Solution.

$$P(B \mid A) = \frac{P(BA)}{P(A)} = \frac{P(A)}{P(A)} = \boxed{1}$$

(c) Find $P(B \mid A^c)$

6 pts

Solution.

$$P(B \mid A^c) = \frac{P(BA^c)}{P(A^c)} = \frac{P(B) - P(A)}{1 - P(A)} = \frac{1/3 - 1/4}{1 - 1/4} = \boxed{\frac{1}{9}}$$

(d) Find $P(B^c \mid A^c)$

6 pts

Solution.

$$P(B^c \mid A^c) = 1 - P(B \mid A^c) = 1 - \frac{1}{9} = \boxed{\frac{8}{9}}$$

(e) Are A and B^c independent? Justify your answer.

6 pts

Solution. Since $A \subset B$, we have $A \cap B^c = \emptyset$, so $P(AB^c) = 0$. On the other hand, $P(A)P(B^c) = (1/4)(2/3) \neq 0$, so $P(A)P(B^c) \neq P(AB^c)$. Hence A and B^c are not independent.

3. Suppose you roll 10 **four-faced** dice, with faces labeled $1, 2, 3, 4$, and each equally likely to appear on top. Let X_1 denote the number showing up on the first die, X_2 the number of the second die, etc., and assume the X_i's are independent.

(a) Find $E\left(\frac{1}{X_1}\right)$.

10 pts

Solution. Using the formula $E(g(X)) = \sum_{x_i} g(x_i)p(x_i)$, we get

$$E\left(\frac{1}{X_1}\right) = \frac{1}{1} \cdot \frac{1}{4} + \frac{1}{2} \cdot \frac{1}{4} + \frac{1}{3} \cdot \frac{1}{4} + \frac{1}{4} \cdot \frac{1}{4} = \boxed{\frac{25}{12}}$$

(b) Let $Z = \min(X_1, X_2, \ldots, X_{10})$ denote the minimum (smallest) of the 10 numbers obtained. Find $P(Z = 2)$, the probability that the smallest of the numbers is 2.

10 pts

Solution. We express this probability as $P(Z = 2) = P(Z \leq 2) - P(Z \leq 1)$ and use the min/max trick, to compute $P(Z \leq 2)$ and $P(Z \leq 1)$:

$$\begin{aligned}
P(Z \leq 2) &= P(\min(X_1, \ldots, X_{10} \leq 2) \\
&= 1 - P(\min(X_1, \ldots, X_{10}) > 2) \\
&= 1 - P(X_1 \geq 3) \ldots P(X_{10} > 2) \\
&= 1 - (1/2)^{10}, \\
P(Z \leq 1) &= 1 - (3/4)^{10}, \\
P(Z = 2) &= P(Z \leq 2) - P(Z \leq 1) = (3/4)^{10} - (1/2)^{10} = \boxed{\frac{3^{10} - 2^{10}}{4^{10}}}
\end{aligned}$$

(c) Let $S = X_1 + \cdots + X_{10}$ denote the sum of the 10 numbers obtained. Find the moment-generating function, $M_S(t)$, of S.

10 pts

Solution. We first compute the mgf of an individual X_i:

$$\begin{aligned}
M_{X_i}(t) = E(e^{tX}) &= \sum_{k=1}^{4} e^{tk} P(X = k) = \frac{1}{4} e^t \sum_{k=0}^{3} e^{tk} \\
&= \frac{e^t(1 - e^{4t})}{4(1 - e^t)},
\end{aligned}$$

using the finite geometric series formula $\sum_{k=0}^{b} r^k = \frac{1-r^{b+1}}{1-r}$. Since the mgf of a sum of independent rv's is the product of the individual mgf's, we get

$$M_S(t) = \prod_1^{10} M_{X_i}(t) = \boxed{\left(\frac{e^t - e^{5t}}{4(1 - e^t)}\right)^{10}}$$

4. Let X be a random variable with density (p.d.f.)

$$f(x) = xe^{-x^2/2}, \quad 0 < x < \infty.$$

(a) Find $P(X \le 4)$.

10 pts

Solution.

$$P(X \le 4) = \int_0^4 xe^{-x^2/2} dx = \int_0^8 e^{-u} du = \boxed{1 - e^{-8}}$$

(b) Find $E(\frac{1}{X})$.

10 pts

Solution.

$$\begin{aligned} E(1/X) &= \int_0^\infty \frac{1}{x} xe^{-x^2/2} dx \\ &= \int_0^\infty e^{-x^2/2} dx \\ &= \sqrt{2\pi} \int_0^\infty (1/\sqrt{2\pi}) e^{-x^2/2}\, dx \\ &= \sqrt{2\pi}(\Phi(\infty) - \Phi(0)) = \sqrt{2\pi}(1 - 0.5) = \boxed{\sqrt{\pi/2}} \end{aligned}$$

(c) Let $Y = X^2$. Find the p.d.f. of Y.

10 pts

Solution. We use the change of variables technique, computing first the c.d.f.'s of X and Y, then differentiating the latter to get the p.d.f. of Y.

$$\begin{aligned} F_Y(y) &= P(Y \le y) = P(X \le \sqrt{y}) = F_X(\sqrt{y}), \\ f_Y(y) &= \frac{d}{dy} F_X(\sqrt{y}) = F_X'(\sqrt{y}) \frac{1}{2\sqrt{y}} \\ &= \sqrt{y} e^{-y/2} \frac{1}{2\sqrt{y}} = \boxed{\frac{1}{2} e^{-y/2}, \quad 0 < y < \infty} \end{aligned}$$

5. Suppose X and Y are discrete random variables with values $1, 2, 3$ each and joint p.m.f. given by

$$f(x, y) = \begin{cases} 1/9 & \text{if } x = y \\ 2/9 & \text{if } x < y \\ 0 & \text{if } x > y \end{cases}$$

for $x, y = 1, 2, 3$.

(a) Find the marginal p.m.f.'s of X and Y.

10 pts

Solution. The matrix representation of the joint distribution is as follows, with marginal distributions of X and Y given in the last column and last row.

$X \setminus Y$	1	2	3	$p_X(x)$
1	1/9	2/9	2/9	5/9
2	0	1/9	2/9	3/9
3	0	0	1/9	1/9
$p_Y(y)$	1/9	3/9	5/9	

(b) Find $E(XY)$.

10 pts

Solution.

$$E(XY) = \sum xyP(x,y) = \frac{1}{9}(1 \cdot 1 + 2 \cdot 2 + 3 \cdot 3) + \frac{2}{9}(1 \cdot 2 + 1 \cdot 3 + 2 \cdot 3) = \boxed{4}$$

(c) Find the conditional p.m.f. of Y given $X = 1$, and represent it in the form of a distribution table (i.e., a 2-row table with the first row listing the values and the second row the associated probabilities).

10 pts

Solution. The conditional p.m.f. of Y given $X = 1$ is given by

$$p(Y = y \mid X = 1) = \frac{p(1,y)}{p_X(1)}.$$

Using the above values for $p(x,y)$ and $p_X(x)$, we get

y	1	2	3
$P(Y = y \mid X = 1)$	1/5	2/5	2/5

6. Let X and Y be random variables with joint density

$$f(x,y) = x - y + 1, \quad 0 \leq x \leq 1, 0 \leq y \leq 1.$$

10 pts

(a) Find the probability that $X + Y \geq 1/2$.

Solution. (This appeared on the Double Integral handout.) The probability is given by the double integral over the above density function over the part of the unit square on which $x + y \geq 0.5$:

$$\begin{aligned}
&\int_{x=0}^{0.5}\int_{y=0.5-x}^{1}(x-y+1)dydx + \int_{x=0.5}^{1}\int_{y=0}^{1}(x-y+1)dydx \\
&= \int_{x=0}^{0.5}\left(xy - \frac{1}{2}y^2 + y\right)\Big|_{y=0.5-x}^{1} dx + \int_{x=0.5}^{1}\left(xy - \frac{1}{2}y^2 + y\right)\Big|_{y=0}^{1} dx \\
&= \int_{0}^{0.5}\left(x(1-\frac{1}{2}+x) - \frac{1}{2}(1-(\frac{1}{2}-x)^2) + (1-\frac{1}{2}+x)\right) dx \\
&\quad + \int_{0.5}^{1}\left(x+\frac{1}{2}\right) dx \\
&= \int_{0}^{0.5}\left(\frac{1}{8} + x + \frac{3}{2}x^2\right) dx + \left(\frac{1}{2}x^2 + \frac{1}{2}x\right)\Big|_{0.5}^{1} \\
&= \frac{1}{2}\cdot\frac{1}{8} + \frac{1}{2\cdot 2^2} + \frac{1}{3\cdot 2^3}\cdot\frac{3}{2} + \frac{3}{8} + \frac{1}{4} = \boxed{\frac{7}{8}}
\end{aligned}$$

10 pts

(b) Find the conditional probability that $Y \geq 1/4$, given that $X = 1/4$.

Solution. The probability to be computed is $\int_{y=1/4}^{1} f_{Y|X}(y|1/4)dy$ or, equivalently, $1-\int_{y=0}^{1/4} f_{Y|X}(y|1/4)dy$, where $f_{Y|X}(y|x)$ is the conditional density of Y given that $X = x$. Now, $f_{Y|X}(y|x) = f(x,y)/f_X(x)$, and

$$f_X(x) = \int_{y=0}^{1}(x-y+1)dy = \left[xy - \frac{y^2}{2} + y\right]_{y=0}^{1} = x + 0.5, \quad 0 \leq x \leq 1,$$

so $f_X(1/4) = 3/4$ and

$$f_{Y|X}(y|1/4) = \frac{f(1/4,y)}{f_X(1/4)} = \frac{1/4 - y + 1}{3/4} = \frac{5}{3} - \frac{4}{3}y, \quad 0 \leq y \leq 1.$$

and

$$\begin{aligned}
P(Y \geq 1/4 | X = 1/4) &= 1 - \int_{0}^{1/4}\left(\frac{5}{3} - \frac{4}{3}y\right) dy \\
&= 1 - \left[\frac{5}{3}y - \frac{4}{3}\frac{y^2}{2}\right]_{y=0}^{1/4} \\
&= 1 - \frac{5}{12} + \frac{1}{24} = \boxed{\frac{5}{8}}
\end{aligned}$$

10 pts

(c) Find $\text{Cov}(X, Y)$.

Solution.

$$\begin{aligned}
E(XY) &= \int_0^1 \int_0^1 (x^2y - xy^2 + xy)dxdy \\
&= \frac{1}{3}\cdot\frac{1}{2} - \frac{1}{3}\cdot\frac{1}{2} + \frac{1}{2}\cdot\frac{1}{2} = \frac{1}{4}, \\
E(X) &= \int_0^1 \int_0^1 (x^2 - xy + x)dxdy = \frac{1}{3} - \frac{1}{2}\cdot\frac{1}{2} + \frac{1}{2} = \frac{7}{12}, \\
E(Y) &= \int_0^1 \int_0^1 (xy - y^2 + y)dxdy = \frac{1}{2}\cdot\frac{1}{2} - \frac{1}{3} + \frac{1}{2} = \frac{5}{12}, \\
\text{Cov}(X, Y) = E(XY) - E(X)E(Y) &= \frac{1}{4} - \frac{7}{12}\cdot\frac{5}{12} = \boxed{\frac{1}{144}}
\end{aligned}$$

7. Suppose X_1 and X_2 are independent random variables, each having density

$$f(x) = \begin{cases} 3x^{-4} & \text{for } 1 < x < \infty, \\ 0 & \text{otherwise.} \end{cases}$$

10 pts

(a) Find $\text{Var}(X_1)$, $\text{Var}(X_2)$, and $\text{Cov}(X_1, X_2)$. (Hint: This requires only a minimal amount of calculations.)

Solution.

$$\begin{aligned}
E(X_1) &= \int_1^\infty x \cdot 3x^{-4}dx = \int_1^\infty 3x^{-3}dx = \frac{3}{2}, \\
E(X_1^2) &= \int_1^\infty x^2 \cdot 3x^{-4}dx = \int_1^\infty 3x^{-2}dx = 3, \\
\text{Var}(X_1) = E(X_1^2) - E(X_1)^2 &= 3 - \left(\frac{3}{2}\right)^2 = \boxed{\frac{3}{4}}
\end{aligned}$$

Since X_2 has the same distribution as X_1, we have $\boxed{\text{Var}(X_2) = \text{Var}(X_1) = 3/4}$.

Since X_1 and X_2 are independent, we have $\boxed{\text{Cov}(X_1, X_2) = 0}$.

10 pts

(b) Let $Z = \max(X_1, X_2)$ denote the larger (maximum) of the two random variables. Find the p.d.f. (density) of Z.

Solution. To find the p.d.f. $f_Z(z)$ of Z, we first compute the c.d.f. $F_Z(z)$, using the "maximum trick" that came up in class on several occasions: For $z \geq 1$,

$$\begin{aligned}
F_Z(z) &= P(Z \leq z) = P(\max(X_1, X_2) \leq z) \\
&= P(X_1 \leq z, X_2 \leq z) = P(X_1 \leq z)P(X_2 \leq z) = F(z)^2.
\end{aligned}$$

Now, $f(z) = 3z^{-4}$ for $z \geq 1$, so $F(z) = \int_1^z 3t^{-4}dt = 1 - z^{-3}$, and therefore

$$\begin{aligned}
F_Z(z) &= (1 - z^{-3})^2, \\
f_Z(z) &= F_Z'(z) = 2(1 - z^{-3})(-(-3)z^{-4}) \\
&= \boxed{6z^{-4} - 6z^{-7} \quad (z \geq 1)}.
\end{aligned}$$

(c) Let $Q = X_2/X_1$. For $q \geq 1$, find $P(Q \geq q)$.

10 pts

Solution. By the independence of X_1 and X_2 and the given density, the joint density is

$$f(x_1, x_2) = (3x_1^{-4})(3x_2^{-4}) = 9x_1^{-4}x_2^{-4}, \quad 1 < x_1, x_2 < \infty.$$

Hence

$$P(Q \geq q) = P(X_2 > qX_1) = \int_{x_1=1}^{\infty} \int_{x_2=qx_1}^{\infty} 9x_1^{-4}x_2^{-4} dx_2 dx_1 = \int_{x_1=1}^{\infty} 9x_1^{-4} \left[\frac{x_2^{-3}}{-3}\right]_{x_2=qx_1}^{\infty} dx_1$$

$$= \int_{x_1=1}^{\infty} 3x_1^{-4}(qx_1)^{-3} dx = \frac{3}{q^3} \int_{x_1=1}^{\infty} x_1^{-7} dx = \boxed{\frac{1}{2q^3}}$$

8. Assume the math scores on the SAT test are normally distributed with mean 500 and standard deviation 60, and the verbal scores are normally distributed with mean 450 and standard deviation 80. (In particular, under this assumption the scores take on a continuum of values and are not restricted to integer values.)

The following problems are independent of each other.

(a) Write down the density function (p.d.f.) of the math score of a randomly chosen student. (The answer should be an explicit elementary function of x, not an expression involving Φ.)

10 pts

Solution. The density of a general normal distribution with parameters μ and σ is $f(x) = (1/\sqrt{2\pi}\sigma)e^{-(x-\mu)^2/(2\sigma^2)}$. Here $\mu = 500$, $\sigma = 60$, so

$$f(x) = \boxed{\frac{1}{\sqrt{2\pi} \cdot 60} e^{-\frac{1}{2}\left(\frac{x-500}{60}\right)^2}, \quad -\infty < x < \infty.}$$

(b) Find the probability a randomly chosen student's *total* score (i.e., the sum of math and verbal scores) is between 1000 and 1100. (Assume independence of the math and verbal scores.) *Leave the answer in terms of the Φ-function, e.g.,* $\Phi(1100) - \Phi(1000)$.

10 pts

Solution. Let X and Y denote the math and verbal scores of the student. Then $X + Y$ is normal $N(500 + 450, 60^2 + 80^2) = N(950, 100^2)$, so

$$P(1000 < X + Y < 1100) = P\left(\frac{1000 - 950}{100} < \frac{X + Y - 950}{100} < \frac{1100 - 950}{100}\right)$$

$$= \boxed{\Phi(1.5) - \Phi(0.5)}$$

Comment: For a sum of independent r.v.'s the *variances*, not the *standard deviations* add up. The standard deviation of the sum is *not* the sum of the standard deviations (i.e., not equal to $60 + 80$, or 140); to get the correct standard deviation one has to first compute the variance of the sum, $60^2 + 80^2$, then take the square root, $\sqrt{60^2 + 80^2} = 100$.

(c) Suppose two students who took both tests are chosen at random. What is the probability that the first student's math score exceeds the second student's verbal score? (Assume independence of the two scores.) *Again, leave the answer in terms of the Φ-function.*

10 pts

Solution. Let X and Y denote the scores of the two students. Then $X - Y$ is $N(500-450, 60^2 + 80^2) = N(50, 100^2)$, so

$$P(X > Y) = P(X - Y > 0) = P\left(\frac{X - Y - 50}{100} > \frac{0 - 50}{100}\right) = P(Z > -1/2)$$

$$\approx 1 - \Phi(-1/2) = \boxed{\Phi(0.5)}$$

9. The following problems are independent of each other.

(a) A computer generates 48 random real numbers, rounds each number to the nearest integer and then computes the average of these 48 rounded values. Assume that the numbers generated are independent of each other and that the rounding errors are distributed uniformly on the interval $[-0.5, 0.5]$. Find the approximate probability that the average of the rounded values is within 0.05 of the average of the exact numbers. *Leave the answer in terms of the* Φ *function, e.g.* $\Phi(\sqrt{48 \cdot 0.05})$.

15 pts

Solution. Let $X_1, \dots, X_{48}$ denote the 48 rounding errors, and $\overline{X} = (1/48) \sum_{i=1}^{48} X_i$ their average. We need to compute $P(|\overline{X}| \leq 0.05)$. Since a rounding error is uniformly distributed on $[-0.5, 0.5]$, its mean is $\mu = 0$ and its variance is $\sigma = \int_{-0.5}^{0.5} x^2 dx = [x^3/3]_{-0.5}^{0.5} = 1/12$. By the Central Limit Theorem, $\overline{X}$ has approximate distribution $N(\mu, \sigma^2/n) = N(0, (1/12)/48) = N(0, 1/24^2)$. Thus $24\overline{X}$ is approximately standard normal, so

$$\begin{aligned} P(|\overline{X}| \leq 0.05) &\approx P(24 \cdot (-0.05) \leq 24\overline{X} \leq 24 \cdot 0.05) \\ &= \Phi(1.2) - \Phi(-1.2) = \boxed{2\Phi(1.2) - 1}. \end{aligned}$$

(b) A lamp requires a particular type of light bulb that has a lifetime that is normally distributed with mean 3 (months) and variance 1. Suppose you that, as soon as a bulb burns out, it is replaced with a new one. What is the smallest number of bulbs you need to purchase so that, with probability at least 0.9772, the lamp burns for at least 40 months? (Give a numerical answer. Note that $0.9772 = \Phi(2)$; you do not need any other normal distribution values.)

15 pts

Solution. Let n be the (unknown) number of light bulbs to be purchased, $X_1, \dots, X_n$ their respective lifetimes, and $S_n = \sum_{i=1}^n X_i$ the total lifetime of all n bulbs. We need to choose n minimal so that $P(S_n \geq 40) \geq 0.9772$. Now, by the CLT,

$$P(S_n \geq 40) \approx P\left(\frac{S_n - 3n}{1\sqrt{n}} \geq \frac{40 - 3n}{1\sqrt{n}} \right) \approx 1 - \Phi\left(\frac{40 - 3n}{1\sqrt{n}} \right)$$

This is equal to 0.9772 when $(40 - 3n)/\sqrt{n} = -2$, or equivalently, $3n - 2\sqrt{n} - 40 = 0$. Setting $x = \sqrt{n}$, the latter equation becomes $3x^2 - 2x - 40 = 0$. Solving (ignoring the negative solution) gives $x = (1/6)(2 + \sqrt{4 + 480} = 4$, so $\boxed{n = x^2 = 16}$ is the number sought.

10. The following problems are independent of each other.

(a) Let $X_1, X_2, X_3, \dots$ be i.i.d. random variables, with mean $\mu = \mathrm{E}(X_i)$ and variance $\sigma^2 = \mathrm{Var}(X_i)$, and let $S_n = \sum_{i=1}^n X_i$ denote the partial sums of the X_i. Given this set-up and notation, state the *Weak Law of Large Numbers* in precise mathematical form, using proper mathematical notation, and including any hypotheses/quantifiers necessary in the statement.

10 pts

Solution. The WLLN states the following: *For any* $\epsilon > 0$,

$$\lim_{n \to \infty} P\left(\left| \frac{S_n}{n} - \mu \right| \leq \epsilon \right) = 1.$$

Note: The italicized phrase is an essential part of the statement of the WLLN.

(b) If X is a random variable with mean 50 and variance 25, what can be said about the probability that X is between 40 and 60? (E.g., how large, or how small, must this probability be, given the above information?)

10 pts

Solution. [This is Example 2a in 8.2.] By Chebychev's inequality,

$$P(|X - 50| > 10) = P(|X - 50| > 5 \cdot 2) \leq \frac{1}{2^2},$$

so

$$P(40 \leq X \leq 60) = 1 - P(|X - 50| > 10) \geq 1 - \frac{1}{4} = \boxed{\frac{3}{4}}$$

11. (Extra Credit) How many 10 letter words are there that contain *each* of the letters M,A,T,H at least once? (Example: M U M Z Z T O P H A) As usual, assume there are 26 letters in the alphabet, and count only words with upper case letters. The formula should be a simple expression, involving no more than a few terms, not a messy summation. (No hints will be given on this problem.)

10 pts

Solution. Let S denote the set of all 10 letter words, let M denote the set of such words containing an M, and define similarly A, T, H. Then we need to compute $\#(M \cap A \cap T \cap H)$, the number of elements in the intersection of these four sets. By De Morgan's Law,

$$\#(M \cap A \cap T \cap H) = \#S - \#(M^c \cup A^c \cup T^c \cup H^c).$$

Now, apply inclusion/exclusion to the latter expression. This converts this expression into one involving intersections of 1, 2, 3, or 4 of the sets M^c, A^c, T^c, H^c. To count these, argue as follows:

- There are 26^{10} 10-letter words altogether, so $\#S = 26^{10}$.
- There are 25^{10} such words that do **not** contain a single "forbidden" letter, so each of the $\binom{4}{1}$ sets M^c, T^c, H^c has 25^{10} elements.
- There are 24^{10} such words that do **not** contain two "forbidden" letters, so each of the $\binom{4}{2}$ pairwise intersections of these 4 sets has 24^{10} elements.
- There are 23^{10} such words that do **not** contain three "forbidden" letters, so each of the $\binom{4}{3}$ triple intersections of these 4 sets has 23^{10} elements.
- There are 22^{10} such words that do **not** contain four "forbidden" letters, so the intersection of all 4 of these sets has 22^{10} elements.

Thus, with inclusion/exclusion the above count becomes

$$26^{10} - \binom{4}{1}25^{10} + \binom{4}{2}24^{10} - \binom{4}{3}23^{10} + \binom{4}{4}22^{10}.$$