Math 461 B/C, Spring 2009
Midterm Exam 1 Solutions and Comments

1. Suppose A, B and C are events with $P(A) = P(B) = P(C) = 1/3$, $P(AB) = P(AC) = P(BC) = 1/4$ and $P(ABC) = 1/5$. For each of the following events, first express the event in set-theoretic notation (e.g., $(A \cup B^c) \cap C$), then find its probability.

   (a) *Neither A nor B* occurs.

10 pts

Solution. The event is $\boxed{A^cB^c}$ or $\boxed{(A \cup B)^c}$. Its probability is given by

$$\begin{aligned} P(A^cB^c) &= P((A \cup B)^c) = 1 - P(A \cup B) \\ &= 1 - P(A) - P(B) + P(AB) = 1 - 2(1/3) + (1/4) = \boxed{\frac{7}{12}} \end{aligned}$$

   (b) *At least one* of the three events A, B, and C occurs.

10 pts

Solution. The event is $\boxed{A \cup B \cup C}$. By inclusion/exclusion,

$$\begin{aligned} P(A \cup B \cup C) &= P(A) + P(B) + P(C) \\ &\quad - P(AB) - P(AC) - P(BC) \\ &\quad + P(ABC) \\ &= 3 \cdot (1/3) - 3 \cdot (1/4) + (1/5) = \boxed{\frac{9}{20}} \end{aligned}$$

   (c) *Exactly two* of the three events A, B, and C occur.

10 pts

Solution. The event is $\boxed{ABC^c \cup AB^cC \cup A^cBC}$. Now

$$\begin{aligned} P(ABC^c) &= P(AB) - P(ABC) = 1/4 - 1/5 = 1/20, \\ P(AB^cC) &= P(AC) - P(ABC) = 1/4 - 1/5 = 1/20, \\ P(A^cBC) &= P(BC) - P(ABC) = 1/4 - 1/5 = 1/20, \end{aligned}$$

and since the three sets ABC^c, AB^cC, A^cBC are disjoint, their probabilities add up, so the given event has probability

$$P(ABC^c \cup AB^cC \cup A^cBC) = 3(1/20) = \boxed{\frac{3}{20}}$$

2. For this problem, an 8-digit number is any string of 8 decimal digits (i.e., digits $0, 1, \ldots, 9$). Leading 0's are allowed, so that, for example, 05435199 is an 8-digit number in this sense.

(a) How many 8-digit numbers consist of exactly three 1's, three 2's, and two 3's? (Example: 12331221)

10 pts

Solution. This is a standard word-counting problem, involving three "letters" (1, 2, 3), with multiplicities 3, 3, and 2, respectively. The number of such "words" is given by the number of permutations of 8 *distinct* objects, namely, 8!, divided by
the number of ways the repeated letters can be permuted, namely 3!, 3! and 2!:

$$\frac{8!}{3!3!2!} \ (= 560)$$

(b) How many 8-digit numbers consist of exactly 4 distinct odd digits (i.e., digits $1, 3, 5, 7, 9$) and 4 distinct even digits (i.e., digits $0, 2, 4, 6, 8$)? (Example: 30497612 consists of the odd digits $1, 3, 7, 9$ and the even digits $0, 2, 4, 6$.)

10 pts

Solution. Method 1: First choose 4 of the 5 odd digits ($\binom{5}{4}$ choices), then choose 4 of the 5 even digits ($\binom{5}{4}$ choices), finally arrange the 8 digits chosen in some order (8! choices), to get

$$\binom{5}{4}\binom{5}{4}8! \ (= 1008000)$$

Method 2: (This method is more complicated and not recommended.) First pick an *ordered* arrangement for 4 distinct odd digits ($5 \cdot 4 \cdot 3 \cdot 2$ choices), then pick an *ordered* arrangement for 4 distinct even digits ($5 \cdot 4 \cdot 3 \cdot 2$ choices), finally pick an E/O pattern consisting of 4 E's and 4 O's ($8!/(4!4!) = \binom{8}{4}$ choices), to get

$$(5 \cdot 4 \cdot 3 \cdot 2)^2 \frac{8!}{4!4!} \ (= 1008000),$$

which is the same answer as before.

Comments:

(1) In the first method, the factor 8! is crucial since it accounts for the different ways the chosen digits can be ordered. (Note that the order clearly has to be taken into account: for example, 30497612 and 40397612 contain the same sets of digits, but are obviously different integers.)

(2) In the second method, the factor 8!/(4!4!) is crucial, since it represents the different even-odd patterns. The factor $(5 \cdot 4 \cdot 3 \cdot 2)^2$ alone only gives the number of choices that fit a *fixed* such pattern (e.g., EOEOEOEO or EEEEOOOO). However, in contrast to some other problems (e.g., the 5 couples problem, where men and women were required to alternate), in this problem there is no restriction on the even/odd patterns.

(c) How many 8-digit numbers contain exactly 2 distinct digits? (Example: 31113131 contains exactly two distinct digits, namely 1 and 3.)

15 pts

Solution. [A problem of the same type was worked in class.] There are $\binom{10}{2}$ ways to choose the two digits. Once the two digits have been chosen, there are 2^8 8-digit numbers that can be formed with these digits. Of these, 2 consist of only one of the digits, leaving $2^8 - 2$ numbers that contain both of the two digits. Thus, the total count is

$$\binom{10}{2}(2^8 - 2) \ (= 11430)$$

Comments: The factor $2^8 - 2$ is equal to $(*)$ $\binom{8}{1} + \binom{8}{2} \cdots + \binom{8}{8}$, as can be seen by expanding $2^8 = (1+1)^8$ by the binomial theorem. In general, as pointed out on the cover sheet, answers should not involve summations with more than a handful of terms. In the above case, full credit was given for (otherwise correct) answers containing the summation $(*)$, as one could regard $(*)$ as a "handful of terms" type summation (though this is a bit of a stretch). In more clear-cut cases, such answers would be penalized. Moral: If you arrive at a lengthy summation, you should look for an alternate approach that leads to a simpler answer.

3. For this problem assume the year has 360 days, divided into 12 months, with each month having exactly 30 days. Consider a group of 10 people, each having one of the 360 days as birthday, with each of the 360 days being equally likely to occur as a given person's birthday.

   (a) What is the probability that all 10 birthdays are *distinct and fall into the same month*?

10 pts

Solution. [This is similar to the "final exam problem" worked in class (which asked for the probability that all three finals fall on the same day).]

Encode the possible assignments of birthdays as tuples $(b_1, \dots, b_{10})$, with b_i denoting the birthday of person i in the group, or equivalently, as a 10-letter "birthday word" $b_1 \dots b_{10}$, with each b_i denoting one of the 360 possible birthdays. The total number of such words, and hence the denominator in the probability formulas, is 360^{10}.

Method 1: To count the number of such words in which all 10 birthdays fall within the same month, first pick a month ($\binom{12}{1}$ choices), then within this month pick 10 distinct days taking order into account ($30 \cdot 29 \cdots 21$ choices). This gives a total count of $\binom{12}{1} 30 \cdot 29 \cdots 21$, and a probability

$$\frac{12 \cdot 30 \cdot 29 \cdots 21}{360^{10}}$$

Method 2: First pick a birthday for the first person in the group (360 choices), then pick a birthday for the second person from the same month as that of the first birthday chosen, but different from that day (29 choices), then pick a birthday for the third person from this month, but different from the first two days chosen (28 choices), etc. This gives a total count of $360 \cdot 29 \cdot 28 \cdots 21$ and probability

$$\frac{360 \cdot 29 \cdot 28 \cdots 21}{360^{10}},$$

which is the same as the above answer.

Comments:

(1) Using $\binom{30}{10}$ instead of $30 \cdot 29 \cdots 21$ as count for the choices of days within the month would correspond to counting *unordered* samples and give an answer that is off by a factor 8!. It is crucial that order is taken into account here, since otherwise the counting would be inconsistent with the denominator count, 360^{10}, which represents the total number of *ordered* birthday words. See the comments on Problem 28 in HW 2.

(2) It is clear from the context of the problem that a random birthday assignment can contain repeated birthdays. (In this regard, the problem is different from the "Final Exam Problem" in class, but it is of the same type as the "with replacement" version of Problem 28 in HW 2.) Thus, the total number of outcomes, and hence the denominator in the probabilities, is 360^{10}, and not $360 \cdot 359 \cdots 351$, or $\binom{360}{10}$.

   (b) What is the probability that *no two birthdays fall into the same month*?

10 pts

Solution. [An analogous question also came up in the "final exam problem" worked in class.]

Method 1: First pick an *ordered* sample of 10 of the 12 months ($12 \cdot 11 \cdots 3$ choices), then, for each of these 10 months chosen, pick one of the 30 days in this month (30^{10} choices). This gives a total count of $12 \cdot 11 \cdots 3 \cdot 30^{10}$ and a probability

$$\frac{12 \cdot 11 \cdots 3 \cdot 30^{10}}{360^{10}} = \frac{12 \cdot 11 \cdots 3}{12^{10}} = \frac{12!}{2! 12^{10}}$$

Method 2: Pick a birthday for the first person in the group (360 choices), then pick a birthday for the second person from a different month than that of the first birthday chosen (330 choices), etc. This gives a total count of $360 \cdot 330 \cdot 300 \cdots 90$ and probability

$$\frac{360 \cdot 330 \cdot 300 \cdots 90}{360^{10}}$$

It is easily checked that the two answers are the same.

(c) What is the probability that there are *exactly three (distinct) pairs of matching birthdays* among the 10 people (with the remaining birthdays distinct)? (Example: 2 each have their birthday on February 19, March 19, and April 30, and 1 each has their birthday on April 1, May 11, May 14, July 4.)

15 pts

Solution. [This is similar to the "two pairs" version of the poker problem; see Problems 15(c) and 16(c) from HW 2.] First pick three distinct days for the three "double" birthdays ($\binom{360}{3}$ choices), then pick 4 distinct days from the remaining 357 days for the single birthdays ($\binom{357}{4}$ choices), finally arrange the resulting set of 3 double birthdays and 4 single birthdays into a "birthday word" ($(*)\ \frac{10}{2!2!2!}$ choices). This gives a total count of $\binom{360}{3}\binom{357}{4}\cdot\frac{10!}{2!2!2!}$ and a probability

$$\frac{\binom{360}{3}\cdot\binom{357}{4}\cdot\frac{10!}{2!2!2!}}{360^{10}}=\frac{\frac{360\cdot 359\cdots 354}{3!\,4!}\cdot\frac{10!}{2!2!2!}}{360^{10}}$$

Comments:

(1) The factor $\frac{10}{2!2!2!}$ (or, equivalently, $\binom{10}{2}\binom{8}{2}\binom{6}{2}$) is crucial here since it accounts for the various ways one can arrange the selected birthdays in some order. This factor does not appear in standard poker problems since there one counts (both in $\#S$ and in $\#E$) *unordered* samples, but it does appear in the "poker dice" problem (Problem 16 of HW 2). As explained in class (see also the comments on HW 2 and HW 3), if repetition is allowed (which is the case here since the same birthday can occur more than once), one has no choice but to use *ordered* samples for the counting.

(2) In choosing birthdays for the three pairs, one needs to count *unordered* samples of 3 out of 360, so the count is $\binom{360}{3}$, and not $360\cdot 359\cdot 358$. This is a somewhat subtle issue, but it already came up in poker problems and was pointed out in class in this context. For example, to count poker hands with two pairs, the two values for the pairs can be chosen in $\binom{13}{2}$ ways, not $13\cdot 12$; by contrast, for poker hands with one pair and one triple, the correct count is $13\cdot 12$. The reason is that "two A's and two B's" and "two B's and two A's" are indistinguishable, while "two A's and three B's" and "two B's and three A's" clearly are different.

(3) By the same argument, the correct count for the birthdays for the four "singles" is $\binom{357}{4}$, and not $357\cdot 356\cdot 355\cdot 354$.

4. The Putnam Contest, the "World's Toughest Math Test", is a math competition for undergraduates in the U.S. and Canada. The top 15 students win prizes of $1,000 or more.

   In the 2006 Putnam Contest, 9 of the top 15 spots went to students from MIT, 2 went to students from Harvard, 2 went to students from Princeton, 1 went to a student from Caltech, and 1 went to a student from Waterloo. Assume there were no ties.

   (a) Given the above information, how many finishing orders are possible, if only the colleges that occupy the various spots are taken into account (and not the individual students)?

   10 pts

   Solution. [Except for different numbers and different wording, this is essentially Problem 29 in Chapter 1, which was about finishing orders in a weight lifter contest.] If only the colleges are taken into account, then counting finishing orders amounts to counting 15 letter words formed with 9 M's, 2 H's, 2 P's, 1 C, and 1 W (with the M's corresponding to spots occupied by MIT students, etc.). This is a standard word counting problem, with the count is given by

$$\frac{15!}{9!2!2!} \ (= 900900)$$

   (b) Suppose that, in addition to the above information, it is also known that MIT occupied exactly 3 of the top 5 spots. Given this additional information, how many finishing orders (counted in the same way as above) are possible?

   15 pts

   Solution. [This is of the same type as the second part of the "weight lifter" problem (Problem 29 in Chapter 1), which asked for finishing orders subject to the condition that the U.S. should occupy 2 of the top 3 spots and 1 of the bottom 3 spots.

   Method 1: This approach is pretty straightforward, and it is the one used in the HW solutions to Problem 29. The idea is to first assign all MIT spots, then assign the remaining spots. Since MIT occupies a total of 9 spots among the top 15 and *exactly* 3 of the top 5, it must occupy exactly 6 of the bottom 10 (i.e., those ranked 6 through 15). There are $\binom{5}{3}$ ways to pick the 3 spots MIT occupies among the top 5, and $\binom{10}{6}$ ways to pick the 6 spots MIT occupies among the bottom 10, so the total number of ways MIT can be assigned spots is $\binom{5}{3}\binom{10}{6}$. Once those spots are picked, there are 6 spots left, to be filled by 2 H's, 2 P's, 1 C, and 1 W. The number of ways to fill these spots is 6!/(2!2!), by the usual word counting argument. Hence the total count is

$$\binom{5}{3}\binom{10}{6}\frac{6!}{2!2!} \ (= 378000)$$

   Note: A subtle, but essential, point was that in the given scenario **exactly** 3 of the MIT students placed in the top 5. If one overlooks this and, after assigning 3 of the top 5 slots to MIT students, treats all remaining 12 slots the same and assigns them among the remaining students (6 MIT, 2 Harvard, 2 Princeton, 1 Waterloo, 1 Caltech), one would end up with

$$\binom{5}{3}\frac{12!}{6!2!2!}.$$

   This is not correct, as it would include cases where MIT has 4 or 5 of the top 5 spots.

   Method 2: This approach is much more complicated and fraught with pitfalls, and definitely not recommended. It was attempted by a few students, but almost none succeeded. It is mentioned here mainly to point out the many traps and pitfalls that one is likely to make if one tries that approach.

   1. Start with the 9 MIT students and pick three of these for the 3 MIT spots among the top 5. This can be done in $\binom{9}{3}$ ways, **but** (this is the first of several pitfalls) that count implies that one considers these 9 students distiguishable, contrary to the assumptions in the problem. If we ignore that issue for now and pretend the MIT students are distinguishable (e.g., labeled $M_1, M_2, \ldots, M_9$), one can then proceed as follows:

2. Pick 3 of the top 5 spots for these 3 MIT students. This can be done in $5 \cdot 4 \cdot 3 = 5!/2!$ ways. (It is **not** $\binom{5}{3}$, since to be consistent with our temporary assumption that the M's are distinguishable, we have to take order into account.)
3. Similarly, pick 6 of the bottom 10 spots for remaining 6 MIT students. This can be done in $10 \cdot 9 \cdots 5 = 10!/4!$ ways (**not** $\binom{10}{6}$ for the same reason as before).
4. Then, as in the first method, fill the remaining 6 slots with with 2 H's, 2 P's, 1 C, and 1 W. ($\frac{6!}{2!2!}$ choices).
5. Up until this point the 9 M's have been considered distinguishable (e.g., labeled $M_1, \dots, M_9$). To "undo" this artificial labeling and consider all M's distingishable, divide the total count by the number of ways $M_1, \dots, M_9$ can be permuted, i.e., by 9!.

Putting all these counts together, one gets

$$\frac{1}{9!}\binom{9}{3}\frac{5!}{2!} \cdot \frac{10!}{4!} \cdot \frac{6!}{2!2!}$$

This does not look at all like the answer obtained above, but writing the binomial coefficients in terms of factorials, one can convert this expression to the one above:

$$\begin{aligned} &= \frac{1}{9!} \cdot \frac{9!}{3!6!} \cdot \binom{5}{3} 3! \cdot \binom{10}{6} 6! \cdot \frac{6!}{2!2!} \\ &= \binom{5}{3}\binom{10}{6}\frac{6!}{2!2!} \end{aligned}$$

5. (a) State the three Kolmogorov axioms. Be sure to use proper mathematical notation and state any conditions that are present in the axioms.

15 pts

Solution. [This was a question of the "state formula/theorem/property" variety; the requested formulas and statements (Kolmogorov axioms and binomial theorem) were explicitly mentioned in the exam syllabus.]

Axiom 1. $0 \le P(A) \le 1$ for any event A.

Axiom 2. $P(S) = 1$.

Axiom 3. If $A_1, A_2, \dots$ are *finitely or countably infinitely many* events that are *mutually disjoint*, then $P(A_1 \cup A_2 \cup \cdots) = P(A_1) + P(A_2) + \cdots$.

Comments: The conditions given in italics, and most importantly, the mutual disjointness, are an essential part of the third axiom. Without these conditions, the axioms would only be satisfied in trivial situations and hence would be useless. For example, without the mutual disjointness condition, one could apply (3) with A_1 and A_2 both equal to an arbitrary event A, yielding $P(A) = P(A \cup A) = P(A) + P(A)$ and $P(A) = 0$ for any event A.

(b) State the Binomial Theorem. (Just state the result, making sure to use mathematically correct notation; no proofs required.)

10 pts

Solution. [This was one of the three bullet points in the exam syllabus for Chapter 1.] The Binomial Theorem says that, for any x, y and any positive integer n,

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

Maximal Total Score: 150 points