Practice problems on double integrals

The problems below illustrate the kind of double integrals that frequently arise in probability applications. The first group of questions asks to set up a double integral of a general function $f(x, y)$ over a giving region in the xy-plane. This means writing the integral as an iterated integral of the form $\int_*^* \int_*^* f(x, y)dxdy$ and/or $\int_*^* \int_*^* f(x, y)dydx$, with specific limits in place of the asterisks. To do this, follow the steps above (most importantly, sketch the given region). The remaining questions are evaluations of integrals over concrete functions.

1. Set up a double integral of $f(x, y)$ over the region given by $0 < x < 1, x < y < x + 1$.
   Solution:
   $$\int_{x=0}^{1} \int_{y=x}^{x+1} f(x, y)dydx$$

2. Set up a double integral of $f(x, y)$ over the part of the unit square $0 \le x \le 1, 0 \le y \le 1$, on which $y \le x/2$.
   Solution:
   $$\int_{x=0}^{1} \int_{y=0}^{x/2} f(x, y)dydx \text{ or } \int_{y=0}^{1/2} \int_{x=2y}^{1} f(x, y)dxdy$$

3. Set up a double integral of $f(x, y)$ over the part of the unit square on which $x + y > 0.5$.
   Solution:
   $$\int_{x=0}^{1/2} \int_{y=1/2-x}^{1} f(x, y)dydx + \int_{x=1/2}^{1} \int_{y=0}^{1} f(x, y)dydx$$

4. Set up a double integral of $f(x, y)$ over the part of the unit square on which **both** x and y are greater than 0.5.
   Solution:
   $$\int_{x=1/2}^{1} \int_{y=1/2}^{1} f(x, y)dydx$$

5. Set up a double integral of $f(x, y)$ over the part of the unit square on which **at least one of** x and y is greater than 0.5.
   Solution:
   $$\int_{x=0}^{1/2} \int_{y=1/2}^{1} f(x, y)dydx + \int_{x=1/2}^{1} \int_{y=0}^{1} f(x, y)dydx$$

6. Set up a double integral of $f(x, y)$ over the part of the region given by $0 < x < 50 - y < 50$ on which **both** x and y are greater than 20.
   Solution:
   $$\int_{x=20}^{30} \int_{y=20}^{50-x} f(x, y)dydx$$

7. Set up a double integral of $f(x, y)$ over the set of all points (x, y) in the first quadrant with $|x - y| \le 1$.
   Solution:
   $$\int_{x=0}^{1} \int_{y=0}^{x+1} f(x, y)dydx + \int_{x=1}^{\infty} \int_{y=x-1}^{x+1} f(x, y)dydx$$

8. Evaluate $\iint_R e^{-x-y} dxdy$, where R is the region in the first quadrant in which $x + y \le 1$.
Solution:

$$\int_{x=0}^{1}\int_{y=0}^{1-x} e^{-x}e^{-y}dydx = \int_0^1 e^{-x}(1-e^{-(1-x)})dx$$
$$= \int_0^1 (e^{-x} - e^{-1})dx = 1 - 2e^{-1}.$$

9. Evaluate $\iint_R e^{-x-2y} dxdy$, where R is the region in the first quadrant in which $x \le y$
Solution:

$$\int_{x=0}^{\infty}\int_{y=x}^{\infty} e^{-x-2y}dydx = \int_0^{\infty} \frac{1}{2}e^{-3x}dx = \frac{1}{6}$$

10. Evaluate $\iint_R (x^2 + y^2) dxdy$, where R is the region $0 \le x \le y \le L$
Solution:

$$\int_{y=0}^{L}\int_{x=0}^{y} (x^2+y^2)dydx = \int_{y=0}^{L} \left(\frac{1}{3}x^3 + y^2 x\right)\Big|_{x=0}^{y} dydx$$
$$= \int_0^L \frac{4}{3}y^3 dy = \frac{L^4}{3}.$$

11. Evaluate $\iint_R (x - y + 1) dxdy$, where R is the region inside the unit square in which $x + y \ge 0.5$.
Solution:

$$\int_{x=0}^{0.5}\int_{y=0.5-x}^{1} (x-y+1)dydx + \int_{x=0.5}^{1}\int_{y=0}^{1} (x-y+1)dydx$$
$$= \int_{x=0}^{0.5} \left(xy - \frac{1}{2}y^2 + y\right)\Big|_{y=0.5-x}^{1} dx + \int_{x=0.5}^{1} \left(xy - \frac{1}{2}y^2 + y\right)\Big|_{y=0}^{1} dx$$
$$= \int_0^{0.5} \left(x(1 - \frac{1}{2} + x) - \frac{1}{2}(1 - (\frac{1}{2} - x)^2) + (1 - \frac{1}{2} + x)\right) dx$$
$$+ \int_{0.5}^{1} \left(x + \frac{1}{2}\right) dx$$
$$= \int_0^{0.5} \left(\frac{1}{8} + x + \frac{3}{2}x^2\right) dx + \left(\frac{1}{2}x^2 + \frac{1}{2}x\right)\Big|_{0.5}^{1}$$
$$= \frac{1}{2}\cdot\frac{1}{8} + \frac{1}{2\cdot 2^2} + \frac{1}{3\cdot 2^3}\cdot\frac{3}{2} + \frac{3}{8} + \frac{1}{4} = \frac{7}{8}$$

12. Evaluate $\int_0^1\int_0^1 x\max(x,y)dydx$.
Solution:

$$\int_{x=0}^{1}\int_{y=0}^{x} x^2 dydx + \int_{x=0}^{1}\int_{y=x}^{1} xydydx = \int_0^1 \left(x^3 + x\frac{1-x^2}{2}\right) dx$$
$$= \frac{1}{4} + \frac{1}{2}\left(\frac{1}{2} - \frac{1}{4}\right) = \frac{3}{8}.$$