Tips on doing double integrals

- **Setting up double integrals:** This means writing the integral over a given region (usually described verbally) as an iterated integral of the form $\int_*^* \int_*^* f(x,y)dxdy$ and/or $\int_*^* \int_*^* f(x,y)dydx$ with specific limits in place of the asterisks. This is the part that usually causes the greatest difficulties. Proceed as follows:

  - **Sketch the given region.** The most important piece of advice here. Don't try to work out limits "in the abstract" or by algebraic manipulations; if you do so, you risk making mistakes that could have been avoided or spotted with a simple sketch.
  - Try to "sweep" out the region either vertically or horizontally, and determine the upper and lower bounds on x and y that correspond to this sweep.
  - Using the bounds on x and y obtained from such a sweep, express the region in terms of inequalities of the form $* \le x \le *$, $* \le y \le *$, making sure that (at least) one of the variables has constant limits. A horizontal sweep corresponds to constant limits on y, while a vertical sweeps corresponds to constant limits on x. In most cases, either type of sweep will work, though often one is more convenient or easier to work with than the other.
  - Finally, use the bounds in these inequalities to set up a double integral over the given region R, i.e., express a double integral $\iint_R f(x,y)dxdy$ as an iterated integral in the form $\int_*^* \int_*^* f(x,y)dxdy$ and/or $\int_*^* \int_*^* f(x,y)dydx$ with specific limits in place of the asterisks.
  - In some (rare) cases, you cannot get away with a single such iterated integral, and you have to split R into two (or more) pieces, with each represented in the above form.

- **Note on the order of integration:** The key rule here is that **the outside integral must have constant limits**; an expression like $\int_x^{2x} \int_0^1 f(x,y)dxdy$ does not make sense. Once you have determined bounds on x and y from a vertical or horizontal sweep of the region as above, the order is usually determined since normally only one of the variables will have constant limits, and that variable therefore has to be the one on the outside integral.

- **Advice on notation:** I strongly recommend writing the integration variable involved explicitly under each integral sign, using notations like $\int_{x=0}^1 \int_{y=x}^{2x} xydydx$, $\left[x^2/2y\right]_{y=x}^{2x}$, etc. Without such notational reminders it is easy to forget which of x and y is considered a variable and which is considered a constant in an integration, resulting in mistakes.

- **Some useful formulas:** The following are some frequently occurring integrals. They are easy to evaluate directly, but knowing these formulas saves valuable time in an exam setting. (There are some mild restrictions on the constant c here: In the first and second formulas, the restrictions on c are $c \ne -1$ respectively $c \ne 1$, in order for the fraction on the right to make sense. The third formula holds for $c > 0$, since otherwise the integral would be infinite.)

$$\int_0^1 x^c dx = \frac{1}{c+1}, \quad \int_1^\infty \frac{1}{x^c} dx = \frac{1}{c-1}, \quad \int_0^\infty e^{-cx} dx = \frac{1}{c}, \int_L^\infty e^{-cx} dx = \frac{e^{-cL}}{c} \quad (c > 0)$$

— Practice problems on back of page —

Practice problems on double integrals

The problems below illustrate the kind of double integrals that frequently arise in probability applications. The first group of questions asks to set up a double integral of a general function $f(x, y)$ over a giving region in the xy-plane. This means writing the integral as an iterated integral of the form $\int_*^* \int_*^* f(x, y) dx dy$ and/or $\int_*^* \int_*^* f(x, y) dy dx$, with specific limits in place of the asterisks. To do this, follow the steps above (most importantly, sketch the given region). The remaining questions are evaluations of integrals over concrete functions.

1. Set up a double integral of $f(x, y)$ over the region given by $0 < x < 1, x < y < x + 1$.
2. Set up a double integral of $f(x, y)$ over the part of the unit square $0 \le x \le 1, 0 \le y \le 1$, on which $y \le x/2$.
3. Set up a double integral of $f(x, y)$ over the part of the unit square on which $x + y > 0.5$.
4. Set up a double integral of $f(x, y)$ over the part of the unit square on which **both** x and y are greater than 0.5.
5. Set up a double integral of $f(x, y)$ over the part of the unit square on which **at least one of** x and y is greater than 0.5.
6. Set up a double integral of $f(x, y)$ over the part of the region given by $0 < x < 50 - y < 50$ on which **both** x and y are greater than 20.
7. Set up a double integral of $f(x, y)$ over the set of all points (x, y) in the first quadrant with $|x - y| \le 1$.
8. Evaluate $\iint_R e^{-x-y} dx dy$, where R is the region in the first quadrant in which $x + y \le 1$.
9. Evaluate $\iint_R e^{-x-2y} dx dy$, where R is the region in the first quadrant in which $x \le y$
10. Evaluate $\iint_R (x^2 + y^2) dx dy$, where R is the region $0 \le x \le y \le L$
11. Evaluate $\iint_R (x - y + 1) dx dy$, where R is the region inside the unit square in which $x + y \ge 0.5$.
12. Evaluate $\int_0^1 \int_0^1 x \max(x, y) dy dx$.

[Solutions will be posted on course webpage]