Binomial coefficients

- **Definition:** \( \binom{n}{r} = \frac{n!}{r!(n-r)!} \) (“\( n \) choose \( r \)”).
  (Here \( n = 1, 2, \ldots \) and \( r = 0, 1, \ldots, n \). Note that, by definition, \( 0! = 1 \).)
- **Alternate definition:** \( \binom{n}{r} = \frac{n(n-1)\ldots(n-r+1)}{r!} \).
  (This version is convenient for hand-calculating binomial coefficients.)
- **Symmetry property:** \( \binom{n}{r} = \binom{n}{n-r} \).
- **Special cases:**
  \( \binom{n}{0} = \binom{n}{n} = 1 \), \( \binom{n}{1} = \binom{n}{n-1} = n \).
- **Binomial Theorem:** \( (x+y)^n = \sum_{r=0}^{n} \binom{n}{r} x^r y^{n-r} \).
- **Combinatorial Interpretations:** \( \binom{n}{r} \) represents
  1. the number of ways to select \( r \) objects out of \( n \) given objects (“unordered samples without replacement”);
  2. the number of \( r \)-element subsets of an \( n \)-element set;
  3. the number of \( n \)-letter HT sequences with exactly \( r \) H’s and \( n-r \) T’s;
  4. the coefficient of \( x^r y^{n-r} \) when expanding \( (x+y)^n \) and collecting terms.

Multinomial coefficients

- **Definition:** \( \binom{n}{n_1, n_2, \ldots, n_r} = \frac{n!}{n_1!n_2!\ldots n_r!} \).
  (Here \( n \) and \( n_1, \ldots, n_r \) are nonnegative integers subject to \( \ast \) \( n = n_1 + n_2 + \cdots + n_r \).)
- **Special cases:**
  Case \( r = 2 \):
  \( \binom{n}{n_1, n_2} = \binom{n}{n_1} = \binom{n}{n_2} \) (since \( n_1 + n_2 = n \), and so \( n_2 = n - n_1 \)).
  Case \( r = n, n_1 = \cdots = n_r = 1 \):
  \( \binom{n}{1, 1, \ldots, 1} = n! \).
- **Multinomial Theorem:** \( (x_1 + \cdots + x_r)^n = \sum_{(\ast)} \binom{n}{n_1, n_2, \ldots, n_r} x_1^{n_1} \ldots x_r^{n_r} \), where
  the sum is taken over all tuples \( (n_1, \ldots, n_r) \) of nonnegative integers that add up to \( n \)
  (i.e., satisfy condition \( \ast \) above).
- **Combinatorial Interpretations:** \( \binom{n}{n_1, n_2, \ldots, n_r} \) represents
  1. the number of ways to split \( n \) distinct objects into \( r \) distinct groups, of sizes \( n_1, \ldots, n_r \), respectively. (In the case \( n_1 = \cdots = n_r = 1 \) this is the number of ways to permute all \( n \) objects.)
  2. the number of \( n \)-letter words formed with \( r \) distinct letters, say, \( L_1, \ldots, L_r \), used \( n_1, \ldots, n_r \) times respectively.
  3. the coefficient of \( x_1^{n_1} \ldots x_r^{n_r} \) when expanding \( (x_1 + \cdots + x_r)^n \) and collecting terms.