Binomial coefficients

- **Definition:** $\binom{n}{r} = \frac{n!}{r!(n-r)!}$ ("n choose r").
  (Here $n = 1, 2, \dots$ and $r = 0, 1, \dots, n$. Note that, by definition, $0! = 1$.)
- **Alternate definition:** $\binom{n}{r} = \frac{n(n-1)\dots(n-r+1)}{r!}$.
  (This version is convenient for hand-calculating binomial coefficients.)
- **Symmetry property:** $\binom{n}{r} = \binom{n}{n-r}$
- **Special cases:** $\binom{n}{0} = \binom{n}{n} = 1, \quad \binom{n}{1} = \binom{n}{n-1} = n$
- **Binomial Theorem:** $(x+y)^n = \sum_{r=0}^{n} \binom{n}{r} x^r y^{n-r}$
- **Combinatorial Interpretations:** $\binom{n}{r}$ represents
  1. the number of ways to select r objects out of n given objects ("unordered samples without replacement");
  2. the number of r-element subsets of an n-element set;
  3. the number of n-letter HT sequences with exactly r H's and $n-r$ T's;
  4. the coefficient of $x^r y^{n-r}$ when expanding $(x+y)^n$ and collecting terms.

Multinomial coefficients

- **Definition:** $\binom{n}{n_1, n_2, \dots, n_r} = \frac{n!}{n_1! n_2! \dots n_r!}$.
  (Here n and $n_1, \dots, n_r$ are nonnegative integers subject to $(*)$ $n = n_1 + n_2 + \dots + n_r$.)
- **Special cases:**
  Case $r = 2$: $\binom{n}{n_1, n_2} = \binom{n}{n_1} = \binom{n}{n_2}$ (since $n_1 + n_2 = n$, and so $n_2 = n - n_1$).
  Case $r = n$, $n_1 = \dots = n_r = 1$: $\binom{n}{1, \dots, 1} = n!$
- **Multinomial Theorem:** $(x_1 + \dots + x_r)^n = \sum_{(*)} \binom{n}{n_1, n_2, \dots, n_r} x_1^{n_1} \dots x_r^{n_r}$, where the sum is taken over all tuples $(n_1, \dots, n_r)$ of nonnegative integers that add up to n (i.e., satisfy condition $(*)$ above).
- **Combinatorial Interpretations:** $\binom{n}{n_1, n_2, \dots, n_r}$ represents
  1. the number of ways to split n distinct objects into r distinct groups, of sizes $n_1, \dots, n_r$, respectively. (In the case $n_1 = \dots = n_r = 1$ this is the number of ways to permute all n objects.)
  2. the number of n-letter words formed with r distinct letters, say, $L_1, \dots, L_r$, used $n_1, \dots, n_r$ times respectively.
  3. the coefficient of $x_1^{n_1} \dots x_r^{n_r}$ when expanding $(x_1 + \dots + x_r)^n$ and collecting terms.