

Continued Fraction Approximations to π

Decimal Expansion:

$\pi = 3.14159265358979323846264338327950288419716939937510582097494459230781$
 $6406286208998628034825342117067982148086513282306647093844609550582231$
 $7253594081284811174502841027019385211055596446229489549303820\dots$

Continued Fraction Expansion:

$\pi = [3, 7, 15, 1, 292, 1, 1, 1, 2, 1, 3, 1, 14, 2, 1, 1, 2, 2, 2, 2, 1, 84, 2, 1, 1, 15, 3, 13, 1,$
 $4, 2, 6, 6, 99, 1, 2, 2, 6, 3, 5, 1, 1, 6, 8, 1, 7, 1, 2, 3, 7, 1, 2, 1, 1, 12, 1, 1, 1,$
 $3, 1, 1, 8, 1, 1, 2, 1, 6, 1, 1, 5, 2, 2, 3, 1, 2, 4, 4, 16, 1, 161, 45, 1, 22, 1, 2, 2, 1, \dots]$

Truncated CF	Rational value	Decimal value	Error
[3]	3	3.000000000000000	$-1.41 \cdot 10^{-1}$
[3; 7]	$\frac{22}{7}$	3.14285714285714	$1.23 \cdot 10^{-3}$
[3; 7, 15]	$\frac{333}{106}$	3.14150943396226	$-8.32 \cdot 10^{-5}$
[3; 7, 15, 1]	$\frac{355}{113}$	3.14159292035398	$2.66 \cdot 10^{-7}$
[3; 7, 15, 1, 292]	$\frac{103993}{33102}$	3.14159265301190	$-5.77 \cdot 10^{-10}$
[3; 7, 15, 1, 292, 1]	$\frac{104348}{33215}$	3.14159265392142	$3.31 \cdot 10^{-10}$
[3; 7, 15, 1, 292, 1, 1]	$\frac{208341}{66317}$	3.14159265346744	$-1.22 \cdot 10^{-10}$
[3; 7, 15, 1, 292, 1, 1, 1]	$\frac{312689}{99532}$	3.14159265361894	$2.91 \cdot 10^{-11}$
[3; 7, 15, 1, 292, 1, 1, 1, 2]	$\frac{833719}{265381}$	3.14159265358108	$-8.71 \cdot 10^{-12}$
[3; 7, 15, 1, 292, 1, 1, 1, 2, 1]	$\frac{1146408}{364913}$	3.14159265359140	$1.61 \cdot 10^{-12}$
[3; 7, 15, 1, 292, 1, 1, 1, 2, 1, 3]	$\frac{4272943}{1360120}$	3.14159265358939	$-4.04 \cdot 10^{-13}$
[3; 7, 15, 1, 292, 1, 1, 1, 2, 1, 3, 1]	$\frac{5419351}{1725033}$	3.14159265358982	$2.22 \cdot 10^{-14}$
[3; 7, 15, 1, 292, 1, 1, 1, 2, 1, 3, 1, 14]	$\frac{80143857}{25510582}$	3.14159265358979	$-4.44 \cdot 10^{-16}$

Some observations:

- The errors alternate in sign, i.e., the approximations alternately overshoot and undershoot π .
- Big jumps in denominators (e.g., 113 \rightarrow 33102) occur at large CF “digits”; small jumps (e.g., 106 \rightarrow 113) occur at small CF “digits”.
- There is no obvious pattern to CF digits of π .

Best Rational Approximations to π

This table shows the best rational approximation p/q to π for each denominator q in $1, 2, 3, \dots, 30, 100, 101, \dots, 115$, along with the error in the approximation, $p/q - \pi$. Boxed values in the third column denote new records, i.e., rationals that approximate π better than any rational with smaller denominator. Boxed values in the second column denote rationals arising from the CF expansion of π ; each of these is a new record approximation.

q	$\frac{p}{q}$	$\frac{p}{q} - \pi$
1	$\frac{3}{1}$	-0.141593
2	$\frac{6}{2}$	-0.141593
3	$\frac{9}{3}$	-0.141593
4	$\frac{13}{4}$	0.108407
5	$\frac{16}{5}$	0.0584073
6	$\frac{19}{6}$	0.025074
7	$\frac{22}{7}$	0.00126449
8	$\frac{25}{8}$	-0.0165927
9	$\frac{28}{9}$	-0.0304815
10	$\frac{31}{10}$	-0.0415927
11	$\frac{35}{11}$	0.0402255
12	$\frac{38}{12}$	0.025074
13	$\frac{41}{13}$	0.0122535
14	$\frac{44}{14}$	0.00126449
15	$\frac{47}{15}$	-0.00825932

q	$\frac{p}{q}$	$\frac{p}{q} - \pi$
16	$\frac{50}{16}$	-0.0165927
17	$\frac{53}{17}$	-0.0239456
18	$\frac{57}{18}$	0.025074
19	$\frac{60}{19}$	0.0163021
20	$\frac{63}{20}$	0.00840735
21	$\frac{66}{21}$	0.00126449
22	$\frac{69}{22}$	-0.00522902
23	$\frac{72}{23}$	-0.0111579
24	$\frac{100}{24}$	-0.0165927
25	$\frac{79}{25}$	0.0184073
26	$\frac{82}{26}$	0.0122535
27	$\frac{85}{27}$	0.00655549
28	$\frac{88}{28}$	0.00126449
29	$\frac{91}{29}$	-0.00366162
30	$\frac{94}{30}$	-0.00825932

q	$\frac{p}{q}$	$\frac{p}{q} - \pi$
100	$\frac{314}{100}$	-0.00159265
101	$\frac{317}{101}$	-0.00297879
102	$\frac{320}{102}$	-0.00433775
103	$\frac{324}{103}$	0.00403841
104	$\frac{327}{104}$	0.00263812
105	$\frac{330}{105}$	0.00126449
106	$\frac{333}{106}$	-0.0000832196
107	$\frac{336}{107}$	-0.00140574
108	$\frac{339}{108}$	-0.00270376
109	$\frac{342}{109}$	-0.00397797
110	$\frac{346}{110}$	0.00386189
111	$\frac{349}{111}$	0.00255149
112	$\frac{352}{112}$	0.00126449
113	$\frac{355}{113}$	0.0000002667
114	$\frac{179}{57}$	-0.00124178
115	$\frac{361}{115}$	-0.00246222

Record approximations to π : Each rational in this list is a new record in the sense that it is closer to π than all rationals with smaller denominator. The list includes *all* such records up to denominator 16604. Rationals arising from the CF expansion of π are boxed.

$$\frac{3}{1}, \frac{13}{4}, \frac{16}{5}, \frac{19}{6}, \frac{22}{7}, \frac{179}{57}, \frac{201}{64}, \frac{223}{71}, \frac{245}{78}, \frac{267}{85}, \frac{289}{92}, \frac{311}{99}, \frac{333}{106}, \frac{355}{113}, \frac{52163}{16604}$$

Each CF approximation is a record, but there are records that are not CF approximations. Some records (e.g., $22/7$ and $355/113$) are long lasting: $22/7$ remains a record until the denominator reaches 57, and the $355/113$ record is not broken until denominator 16604.