6 Continued fractions

6.1 Definitions and notations

Definition 6.1 (Continued fractions). A finite or infinite expression of the form

\[
a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \cdots}}
\]

where the \(a_i\) are real numbers, with \(a_1, a_2, \ldots > 0\), is called a continued fraction (c.f.). The numbers \(a_i\) are called the partial quotients of the c.f.

The continued fraction (6.1) is called simple if the partial quotients \(a_i\) are all integers. It is called finite if it terminates, i.e., if it is of the form

\[
a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \cdots + \frac{1}{a_n}}}
\]

and infinite otherwise.

Notation (Bracket notation for continued fractions). The continued fractions (6.1) and (6.2) are denoted by \([a_0, a_1, a_2, \ldots]\) and \([a_0, a_1, a_2, \ldots, a_n]\), respectively. In particular,

\[
[a_0] = a_0, \quad [a_0, a_1] = a_0 + \frac{1}{a_1}, \quad [a_0, a_1, a_2] = a_0 + \frac{1}{a_1 + \frac{1}{a_2}}, \quad \ldots
\]

Remarks. (i) Note that the first term, \(a_0\), is allowed to be negative or 0, but all subsequent terms \(a_i\) must be positive. This requirement ensures that there are no zero denominators and that any finite c.f. (6.2), and all of its convergents, are well-defined.

(ii) In the sequel we will focus on the case of simple c.f.’s, i.e., c.f.’s where all partial quotients are integers.

6.2 Convergence of infinite continued fractions

Definition 6.2 (Convergents). The convergents of a (finite or infinite) c.f. \([a_0, a_1, a_2, \ldots]\) are defined as

\[
C_0 = [a_0], \quad C_1 = [a_0, a_1], \quad C_2 = [a_0, a_1, a_2], \ldots
\]

If the c.f. is simple, its convergents \(C_i\) represent rational numbers, denoted by

\[
C_i = \frac{p_i}{q_i},
\]

where \(p_i/q_i\) is in reduced form.

Definition 6.3 (Convergence of infinite continued fractions). An infinite c.f. \([a_0, a_1, a_2, \ldots]\) is called convergent if its sequence of convergents \(C_i = [a_0, a_1, \ldots, a_i]\) converges in the usual sense, i.e., if the limit

\[
\alpha = \lim_{i \to \infty} C_i = \lim_{i \to \infty} [a_0, a_1, \ldots, a_i]
\]
exists (and is a real number). In this case, we say that the continued fraction \([a_0, a_1, a_2, \ldots]\) represents the number \(\alpha\), or is a continued fraction expansion of \(\alpha\), and we write
\[
\alpha = [a_0, a_1, a_2, \ldots].
\]

**Theorem 6.4** (Convergence of infinite simple c.f.’s). Any infinite simple c.f. \([a_0, a_1, \ldots]\) is convergent and thus represents some real number.

### 6.3 Properties of Convergents

**Proposition 6.5** (Formulas for \(p_i\) and \(q_i\)). Let \(\alpha = [a_0, a_1, \ldots]\) be a simple c.f. with convergents \(C_i = [a_0, a_1, \ldots, a_i] = \frac{p_i}{q_i}\).

(i) **Recursion formula:** The numbers \(p_i\) and \(q_i\) are given by the recurrence
\[
p_i = a_i p_{i-1} + p_{i-2},
q_i = a_i q_{i-1} + q_{i-2}
\]
for \(i = 1, 2, \ldots\), along with the initial conditions \(p_0 = a_0, p_{-1} = 1, q_0 = 1, q_{-1} = 0\).

(ii) **Matrix representation:** For \(i = 0, 1, 2, \ldots\)
\[
\begin{pmatrix}
 a_0 & 1 \\
 1 & 0
\end{pmatrix} \begin{pmatrix}
 a_1 & 1 \\
 1 & 1
\end{pmatrix} \cdots \begin{pmatrix}
 a_i & 1 \\
 1 & 0
\end{pmatrix} = \begin{pmatrix}
 p_i & p_{i-1} \\
 q_i & q_{i-1}
\end{pmatrix}
\]

**Theorem 6.6** (Properties of convergents). The convergents \(C_i = \frac{p_i}{q_i}\) of an infinite simple continued fraction \(\alpha = [a_0, a_1, a_2, \ldots]\) satisfy:

(i) \((p_i, q_i) = 1\) for \(i = 0, 1, \ldots\); i.e., the fractions \(p_i/q_i\) are reduced.

(ii) \(q_1 < q_2 < \cdots\); i.e., for \(i \geq 1\), the denominators \(q_i\) are strictly increasing.

(iii) \(C_0 < C_2 < C_4 < \cdots < \alpha < \cdots < C_5 < C_3 < C_1\). That is, the even-indexed convergents form an increasing sequence, while the odd-indexed convergents form a decreasing sequence, with the value of the c.f. sandwiched between both sequences.

(iv) \(C_{i+1} - C_i = \frac{(-1)^i}{q_i q_{i+1}}\) for \(i = 0, 1, 2, \ldots\).

(v) \(|\frac{p_i}{q_i} - \alpha| < \frac{1}{q_i q_{i+1}}\) for \(i = 0, 1, 2, \ldots\).

(vi) **Best approximation property:** For any rational number \(a/b\) with \(a \in \mathbb{Z}, b \in \mathbb{N}\), and \(1 \leq b \leq q_i\),
\[
|\frac{p_i}{q_i} - \alpha| \leq \left| \frac{a}{b} - \alpha \right|,
\]
with equality if and only if \(a/b = p_i/q_i\). That is, the convergent \(p_i/q_i\) is the best-possible approximation to \(\alpha\) among all rational numbers with the same or smaller denominator.
6.4 Expansions of real numbers into continued fractions

**Proposition 6.7** (Continued fraction algorithm). Given a real number \( \alpha \), define successively real numbers \( \alpha_0, \alpha_1, \ldots \), and integers \( a_0, a_1, \ldots \) by

\[
\begin{align*}
\alpha_0 &= \alpha, & a_0 &= \lfloor \alpha_0 \rfloor, \\
\alpha_1 &= \frac{1}{\alpha_0 - \lfloor \alpha_0 \rfloor}, & a_1 &= \lfloor \alpha_1 \rfloor, \\
\alpha_2 &= \frac{1}{\alpha_1 - \lfloor \alpha_1 \rfloor}, & a_2 &= \lfloor \alpha_2 \rfloor, \\
& \vdots & \vdots \\
\end{align*}
\]

where \( \lfloor x \rfloor \) denotes the integer part of \( x \) (i.e., the “floor function”). Stop the algorithm if \( \alpha_n \) is an integer (and thus \( a_n = \alpha_n \)); otherwise continue indefinitely. Then \( [a_0, a_1, \ldots] \) is a simple c.f. that represents the number \( \alpha \). Moreover, for any \( i \geq 0 \) we have

\[
\alpha_i = [a_i, a_{i+1}, \ldots], \quad \alpha = [a_0, a_1, \ldots, a_{i-1}, a_i].
\]

**Theorem 6.8** (Continued fraction expansion of rational numbers). Any finite simple c.f. represents a rational number. Conversely, any rational number \( \alpha \) can be expressed as a simple finite c.f. \( \alpha = [a_0, a_1, \ldots, a_n] \). Moreover, under the requirement that \( a_n > 1 \), this representation is unique. Thus, there is a one-to-one correspondence between rational numbers and finite simple c.f.’s with last partial quotient greater than 1.

**Theorem 6.9** (Continued fraction expansion of irrational numbers). Any infinite simple c.f. represents an irrational number. Conversely, any irrational number \( \alpha \) can be expressed as a simple infinite c.f. \( \alpha = [a_0, a_1, a_2, \ldots] \), and this representation is unique. Thus, there is a one-to-one correspondence between irrational numbers and infinite simple c.f.’s.

**Theorem 6.10** (Continued fraction expansion of quadratic irrationals). The c.f. expansion of a quadratic irrational (i.e., a solution of a quadratic equation with integer coefficients) is eventually periodic, i.e., of the form

\[
[a_0, \ldots, a_N, \overline{a_{N+1}, \ldots, a_{N+p}}],
\]

where the bar indicates the periodic part. Conversely, any infinite simple c.f. that is eventually periodic represents a quadratic irrational. Thus, there is a one-to-one correspondence between quadratic irrationals and infinite, eventually periodic simple c.f.’s.