Linear Differential Equations and Matrix Equations

The general theory of linear differential equations, as developed in class and in Section 3.2 of the text, is very much analogous to the (much simpler) theory of matrix equations (or, equivalently, systems of linear equations), described in this handout. Even the proofs are, to a large extent, analogous in both of these contexts. This analogy helps understand and motivate the general principles behind the theory of linear differential equations.

### Linear Differential Equations:

1. \( L[y] = 0 \) (hom. linear DE),
2. \( L[y] = g(t) \) (nonhom. linear DE).

Here \( L[y] \) is a linear differential operator, such as \( L[y] = y'' + p(t)y' + q(t)y \).

The goal is to find all solutions \( y(t) \) to the homogeneous equation (1) or the nonhomogeneous equation (2).

### Matrix Equations:

1. \( Ax = 0 \) (hom. matrix equation),
2. \( Ax = b \) (nonhom. matrix equation).

Here \( A \) is an \( n \times n \) matrix, \( b \) is a given \( n \)-dimensional vector, and \( x \) is the unknown vector we are trying to solve for.

The goal is find all solutions \( x \) to the homogeneous matrix equation (3) or the nonhomogeneous matrix equation (4).

### Solving the Matrix Equations (3) and (4)

The key steps in this process are listed below. Notice that these steps, and the proofs involved, are largely analogous to corresponding steps in solutions to the differential equations (1) and (2), as described in class. Understanding these ideas in the (simpler and more concrete) context of matrix equations helps understand the same ideas in the more abstract setting of linear operators and linear differential equations.

1. **Linearity of Matrix Multiplication:** Given an \( n \times n \) matrix \( A \), the function \( f(x) = Ax \) is a linear function from \( \mathbb{R}^n \) to \( \mathbb{R}^n \), i.e., it preserves linear combinations: For any vectors \( x_1, \ldots, x_k \) in \( \mathbb{R}^n \) and any constants \( c_1, \ldots, c_k \),

   \[
   A(c_1 x_1 + \cdots + c_k x_k) = c_1 A x_1 + \cdots + c_k A x_k.
   \]

   **Proof:** This follows from the properties of matrix multiplication.

2. **Superposition principle for matrix equations:** If \( x_1, \ldots, x_k \) are solutions to (3), then so is any linear combination \( c_1 x_1 + \cdots + c_k x_k \).

   **Proof:** Suppose \( Ax_1 = 0, \ldots, Ax_k = 0 \). Then, by the linearity of matrix multiplication,

   \[
   A(c_1 x_1 + \cdots + c_k x_k) = c_1 (Ax_1) + \cdots + c_k (Ax_k) = c_1 \mathbf{0} + \cdots + c_k \mathbf{0} = \mathbf{0}.
   \]

3. **General solution to homogeneous matrix equation:** The general solution to (3) is of the form

   \[
   x = c_1 x_1 + \cdots + c_k x_k
   \]

   where \( c_1, \ldots, c_k \) are arbitrary constants and \( x_1, \ldots, x_k \) form a fundamental set of solutions.
Proof: This result lies deeper and is proved in linear algebra courses. In linear algebra terminology, the vectors \( x_1, \ldots, x_k \) that form a fundamental set of solutions are a basis of the nullspace of the matrix \( A \). The number \( k \) of these vectors is the dimension of this nullspace. If \( k = 0 \), there is only the trivial solution, \( 0 \); if \( k = 1 \), the solutions are of the form \( c x_1 \), where \( x_1 \) is a nonzero vector and \( c \) is an arbitrary constant; if \( k = 2 \), the solutions are of the form \( c_1 x_1 + c_2 x_2 \), and so on.

4. General solution to nonhomogeneous matrix equation: If \( x_p \) is a particular solution to the nonhomogeneous equation (4), then the general solution to this equation is given by
\[
x = x_p + x_h
\]
where \( x_h \) is the general solution to the corresponding homogeneous equation (3). In other words, we have the following principle:
The general solution to the nonhomogeneous equation (4) is a particular solution to the nonhomogeneous equation plus the general solution to the corresponding homogeneous equation.

Proof: If \( x_h \) is a solution to \( A x = 0 \) and \( x_p \) is a solution to \( A x = b \), then \( A(x_h + x_p) = A x_h + A x_p = 0 + b = b \) (by the linearity of matrix multiplication). Thus, any vector of the form \( x_h + x_p \) is a solution to the nonhomogeneous equation (4). Conversely, suppose \( x \) is a solution to the nonhomogeneous equation \( A x = b \). Let \( x_h = x - x_p \). Then, by linearity, \( A x_h = A(x - x_p) = A x - A x_p = b - b = 0 \), so \( x_h \) is a solution to the homogeneous equation. Thus, any solution \( x \) to the nonhomogeneous equation is of the form \( x = x_h + x_p \), where \( x_h \) is a solution to the corresponding homogeneous equation.

5. Superposition principle for nonhomogeneous matrix equations: If \( x_{p1} \) is a solution to the nonhomogeneous equation \( A x = b_1 \) and \( x_{p2} \) is a solution to the nonhomogeneous equation \( A x = b_2 \), then \( x_{p1} + x_{p2} \) is a solution to \( A x = b_1 + b_2 \).

Proof: Suppose \( A x_{p1} = b_1 \) and \( A x_{p2} = b_2 \). Then \( A(x_{p1} + x_{p2}) = A x_{p1} + A x_{p2} = b_1 + b_2 \).