Math 408, Spring 2008
Final Exam Solutions

12 pts

1. Assume A and B are independent events with $P(A) = 0.2$ and $P(B) = 0.3$. Let C be the event that **none** of the events A and B occurs, let D be the event that **exactly one** of the events A and B occurs.

   (a) Find $P(C)$.
   Solution. $P(C) = P(A' \cap B') = P(A')P(B') = (1 - P(A))(1 - P(B)) = 0.8 \cdot 0.7 = \boxed{0.56}$

   (b) Find $P(D)$.
   Solution. $P(D) = P(A \cup B \setminus A \cap B) = P(A \cup B) - P(A)P(B) = P(A) + P(B) - 2P(A \cap B) = \boxed{0.38}$

   (c) Find $P(A|D)$.
   Solution. $P(A|D) = P(A \cap D)/P(D) = P(A \setminus A \cap B)/P(D) = (0.2 - 0.2 \cdot 0.3)/0.38 = 7/19 = \boxed{0.368}$

   (d) Are C and D independent? Justify your answer!
   Solution. Since the events C ("none of A and B occurs") and D ("exactly one of A and B occurs") are mutually exclusive, we have $P(C \cap D) = 0$. On the other hand, from above $P(C)P(D) = 0.56 \cdot 0.38 \neq 0$, so $P(C \cap D) \neq P(C)P(D)$ and C and D are therefore **not independent**.

5 pts

2. Suppose the number of hurricanes in a given year has Poisson distribution with mean 3, and that the occurrence of hurricanes in any given year is independent of that in any other year. Consider a 2 year period (say, 2009 and 2010). Find the probability that there are **at least** 3 hurricanes in this 2-year period.

   Solution. Let X and Y denote the number of hurricanes in the first and second years. We need to compute $P(X + Y \geq 3)$, or equivalently, $1 - P(X + Y \leq 2)$. The probability $P(X + Y \leq 2)$ breaks down into that of the following 6 cases for (X, Y): $(0, 0), (1, 1), (1, 0), (0, 1), (2, 0), (0, 2)$. Using the independence of X and Y and the formula $P(X = x) = e^{-\lambda}\lambda^x/x!$ for a Poisson distribution with mean λ, we get

$$\begin{aligned} P(X + Y \leq 2) &= P(0, 0) + P(1, 1) + P(0, 1) + P(1, 0) + P(0, 2) + P(2, 0) \\ &= \frac{e^{-3}3^0}{0!}\frac{e^{-3}3^0}{0!} + \frac{e^{-3}3^1}{1!}\frac{e^{-3}3^1}{1!} + 2\frac{e^{-3}3^0}{0!}\frac{e^{-3}3^1}{1!} + 2\frac{e^{-3}3^0}{0!}\frac{e^{-3}3^2}{2!} \\ &= e^{-6}\left(1 + 9 + 2 \cdot 3 + 2 \cdot \frac{9}{2}\right) = 25e^{-6}. \end{aligned}$$

   Thus, $P(X + Y \geq 3) = \boxed{1 - 25e^{-6} = 0.938}$.

5 pts

3. A particular flight has room for 30 passengers. The airline knows from past experience that, on average, only 80% of all passengers holding reservations show up for the flight. Suppose the airline sells 32 tickets for this flight. What is the probability that the flight is overbooked, i.e., that more people show up for the flight than there are seats available? *Give a numerical answer.*

   Solution. Think of the 32 passengers holding reservations as 32 success/failure trials, with success meaning that the passenger shows up for the flight. Since, on average, 80% of all passengers show up, the success probability is $p = 0.8$. The probability we need to compute is that of getting more than 30 successes, which breaks down into the two cases, 31 successes and 32 successes, with total probability

$$\binom{32}{31} 0.8^{31} 0.2^{1} + \binom{32}{32} 0.8^{32} = 32 \cdot 0.8^{31} 0.2 + 0.8^{32} = \boxed{0.00713\ldots}$$

8 pts

4. **(No calculators for this problem.)** Suppose X is a continuous random variable with positive values and density equal to

$$f(x) = 9xe^{-3x}, \quad 0 < x < \infty.$$

   (a) Find $P(X > 1)$.
   Solution. We have

$$\begin{aligned} P(X > 1) &= \int_1^\infty 9xe^{-3x} dx \\ &= \left[-3xe^{-3x}\right]_1^\infty + 3\int_1^\infty e^{-3x} dx \\ &= 3e^{-3} + e^{-3} = \boxed{4e^{-3}} \end{aligned}$$

   (b) Find $E\left(\frac{1}{X}\right)$.
   Solution.

$$E(1/X) = \int_0^\infty (1/x) 9xe^{-3x} dx = \int_0^\infty 9e^{-3x} dx = \boxed{3}$$

8 pts

5. **(No calculators for this problem.)** Let X be a random variable with probability density function

$$f(x) = \begin{cases} x+1 & \text{if } -1 < x < 0, \\ 1-x & \text{if } 0 \le x < 1, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find $E(X^2)$.

Solution. Using the formula $E(X^2) = \int x^2 f(x)$ and splitting the integral at $x = 0$, we get

$$\begin{aligned} E(X^2) &= \int_{-1}^{0} x^2(x+1)dx + \int_{0}^{1} x^2(1-x)dx \\ &= \left[\frac{1}{4}x^4 + \frac{1}{3}x^3\right]_{-1}^{0} + \left[\frac{1}{3}x^3 - \frac{1}{4}x^4\right]_{0}^{1} \\ &= \boxed{\frac{1}{6}} \end{aligned}$$

(b) Find the c.d.f. $F(x)$ of X.

Solution. [This was a homework problem (3.2-10 from HW 6)] Since the range of f is the interval $(-1, 1)$, we have $F(x) = 0$ for $x \le -1$ and $F(x) = 1$ for $x \ge 1$. Thus it remains to consider the case when $-1 < x < 1$. Now, for $-1 < x < 1$, we have $F(x) = \int_{-1}^{x} f(t)dt$, but since the density $f(f)$ is defined differently according to whether $t < 0$ or $t > 0$, we need to break up the integral at $t = 0$, and consider the ranges $-1 < x < 0$ and $0 < x < 1$ separately.

For $-1 < x \le 0$,

$$\begin{aligned} F(x) &= \int_{-1}^{x} f(t)dt = \int_{-1}^{x} (1+t)dt = \left[t + \frac{1}{2}t^2\right]_{-1}^{x} \\ &= x + \frac{1}{2}x^2 - ((-1) + \frac{1}{2}(-1)^2) = x + \frac{1}{2}x^2 + \frac{1}{2}. \end{aligned}$$

For $0 \le x < 1$, we have

$$\begin{aligned} F(x) &= \int_{-1}^{x} f(t)dt = \int_{-1}^{0} f(t)dt + \int_{0}^{x} f(t)dt \\ &= \int_{-1}^{0} (1+t)dt + \int_{0}^{x} (1-t))dt = \frac{1}{2} + x - \frac{1}{2}x^2 = -\frac{1}{2}x^2 + x + \frac{1}{2}. \end{aligned}$$

Combining all four cases, we get

$$F(x) = \begin{cases} 0 & \text{if } x \le -1, \\ \frac{1}{2}x^2 + x + \frac{1}{2} & \text{if } -1 < x < 0, \\ -\frac{1}{2}x^2 + x + \frac{1}{2} & \text{if } 0 \le x < 1, \\ 1 & \text{if } x \ge 1. \end{cases}$$

(Note that any c.d.f. is equal to 1 to the right of its range, so for $x > 1$, we have $F(x) = 1$, not $F(x) = 0$.)

5 pts

6. **(No calculators for this problem.)** Let X be a random variable with density (p.d.f.)

$$f(x) = xe^{-x^2/2}, \quad 0 < x < \infty.$$

Find the density of $Y = X^2$.

Solution. [This was a problem from HW 7, Problem 3.5-2 in Hogg/Tanis.] We first compute the c.d.f.'s of X and Y:

$$F_X(x) = \int_0^x te^{-t^2/2}dt = \int_0^{x^2/2} e^{-u}du = 1 - e^{-x^2/2},$$
$$F_Y(y) = P(Y \le y) = P(X \le \sqrt{y}) = F_X(\sqrt{y}) = 1 - e^{-y/2}.$$

Differentiating the latter gives the density function of Y:

$$f_Y(y) = F_Y'(y) = \boxed{\frac{1}{2}e^{-y/2}, \quad y > 0}.$$

8 pts

7. **(No calculators for this problem.)** The following problems are independent. For each problem **set up (but do not evaluate) an integral, or a sum of two integrals, giving the requested probability.** Your integral(s) should in one of the following forms (you decide which), with **concrete** numbers or functions in place of the asterisks:

$$\int_{x=*}^{x=*} \int_{y=*}^{y=*} \star \, dy \, dx, \qquad \int_{y=*}^{y=*} \int_{x=*}^{x=*} \star \, dx \, dy.$$

The answer can be either a single expression of this type, or a sum of two such expressions. *Again, leave the integral(s) unevaluated.*

(a) A device requires two components to operate; it stops running as soon as one of the two components fails. The joint density function of the lifetimes of the two components, measured in hours, is

$$f(x,y) = \begin{cases} \frac{1}{27}(x+y) & \text{for } 0 < x < 3 \text{ and } 0 < y < 3, \\ 0 & \text{otherwise.} \end{cases}$$

Set up an integral (or a sum of two integrals) for the probability that **the device fails during its first hour of operation**.

Solution. (Cf. Problem 4-9 from Problemset 4) If we denote the lifetimes of the two components by X and Y, then the device fails during the first hour if $X \leq 1$ *or* $Y \leq 1$, Since the range of the joint density is the 3 by 3 square $0 < x < 3, 0 < y < 3$, this event corresponds to the part of this square in which *at least one of* the variables is ≤ 1. This is an L-shaped region, consisting of the two disjoint rectangles $0 < x \leq 1, 0 < y < 3$ and $1 < x < 3, 0 < y < 1$. Therefore,

$$P(X \leq 1 \text{ or } Y \leq 1) = \int_{x=0}^{1} \int_{y=0}^{3} \frac{1}{27}(x+y)dydx + \int_{x=1}^{3} \int_{y=0}^{1} \frac{1}{27}(x+y)dydx$$

(b) Suppose the future lifetimes (in months) of two components of a machine have joint density function

$$f(x,y) = \begin{cases} c(5-x-y) & \text{if } 0 < x < 5-y < 5, \\ 0 & \text{otherwise,} \end{cases}$$

where c is a constant. Set up an integral (or a sum of two integrals) for the probability that **both components are still functioning two months from now**. (You do not need to evaluate the constant c.)

Solution. (Cf. Problem 4-16 from Problemset 4) A sketch shows that the region described by the inequalities $0 < x < 5 - y < 5$ is a triangle with vertices $(0,0)$, $(5,0)$, and $(0,5)$. The probability sought, that both components are still functioning 2 months from now, corresponds to the part of this triangle in which *both* x and y are greater than 2. A sketch shows that this part is described by the inequalities $2 \leq x \leq 3$, $2 \leq y \leq 5 - x$. Hence, the probability requested is

$$P(X > 2 \text{ and } Y > 2) = \int_{x=2}^{3} \int_{y=2}^{5-x} c(5-x-y)dydx.$$

5 pts

8. Let X and Y be continuous r.v.'s with joint density

$$f(x,y) = \frac{3x^2}{1-x}, \quad 0 < x < 1, x < y < 1,$$

Find $E(Y|1/4)$, the conditional expectation of Y, given $X = 1/4$.

Solution. We have, for $0 \le x \le 1$,

$$\begin{aligned}
f_X(x) &= \int f(x,y)dy = \int_{y=x}^{1} \frac{3x^2}{1-x}\, dx = 3x^2, \\
h(y|x) &= \frac{f(x,y)}{f_X(x)} = \frac{(3x^2)/(1-x)}{3x^2} = \frac{1}{1-x}, \quad x \le y \le 1, \\
E(Y|x) &= \int y h(y|x) dy = \int_{y=x}^{1} y \frac{1}{1-x} dy = \frac{1^2 - x^2}{2(1-x)} = \frac{1+x}{2}. \\
E(Y|1/4) &= \frac{1+1/4}{2} = \boxed{\frac{5}{8} = 0.625}
\end{aligned}$$

5 pts

9. Let X and Y be random variables with joint density

$$f(x,y) = x - y + 1, \quad 0 \le x \le 1, 0 \le y \le 1.$$

Find the probability that $X + Y \ge 1/2$.

Solution. The probability is given by the double integral over the above density function over the part of the unit square on which $x + y \ge 0.5$:

$$\begin{aligned}
&\int_{x=0}^{0.5} \int_{y=0.5-x}^{1} (x - y + 1) dy dx + \int_{x=0.5}^{1} \int_{y=0}^{1} (x - y + 1) dy dx \\
&= \int_{x=0}^{0.5} \left(xy - \frac{1}{2} y^2 + y \right) \Big|_{y=0.5-x}^{1} dx + \int_{x=0.5}^{1} \left(xy - \frac{1}{2} y^2 + y \right) \Big|_{y=0}^{1} dx \\
&= \int_{0}^{0.5} \left(x(1 - \frac{1}{2} + x) - \frac{1}{2}(1 - (\frac{1}{2} - x)^2) + (1 - \frac{1}{2} + x) \right) dx \\
&\quad + \int_{0.5}^{1} \left(x + \frac{1}{2} \right) dx \\
&= \int_{0}^{0.5} \left(\frac{1}{8} + x + \frac{3}{2} x^2 \right) dx + \left(\frac{1}{2} x^2 + \frac{1}{2} x \right) \Big|_{0.5}^{1} \\
&= \frac{1}{2} \cdot \frac{1}{8} + \frac{1}{2 \cdot 2^2} + \frac{1}{3 \cdot 2^3} \cdot \frac{3}{2} + \frac{3}{8} + \frac{1}{4} = \boxed{\frac{7}{8} = .875}
\end{aligned}$$

5 pts

10. A company offers collision insurance and liability insurance. The claim amount under collision insurance is normally distributed with mean 10,000 and standard deviation 2,000; the claim amount under liability insurance is normally distributed with mean 9,000 and standard deviation 2,000. Assuming independence, find the probability that the claim amount under liability insurance exceeds the claim amount under collision insurance.

Solution. We need to compute $P(Y > X)$, or, equivalently, $P(X - Y < 0)$, where X and Y are independent normal r.v.'s with respective means 10,000 and 9,000 and variances $2,000^2$. Now, $X - Y$ has normal distribution with mean $10,000 - 9,000 = 1000$, variance $2 \cdot 2,000^2 = 8,000,000$, and standard deviation $\sqrt{8,000,000} = 2,828$. Converting to standard units, we get

$$\begin{aligned} P(X - Y < 0) &= P\left(\frac{X - Y - 1000}{2828} < \frac{0 - 1000}{2828}\right) \\ &= P(Z < -0.3535) = P(Z > 0.3535) = 1 - P(Z \leq 0.3535) \\ &= 1 - \Phi(0.3535) = \boxed{0.362}. \end{aligned}$$

9 pts

11. **(No calculators for this problem.)** For this problem a "7-digit integer" is any string of 7 digits from $\{0, 1, \ldots, 9\}$. (Thus, for example, 0230917, 9999999 or 0000000 all count). You can and should leave your answers in "raw" form, involving binomial coefficients, factorials, etc. (but no lengthy summations).

(a) What is the probability that a random 7-digit integer has at least one repeated digit?

Solution. This is a birthday (or elevator) type problem. It can only be done using the complement trick. The total number of 7-digit integers is 10^7, and the number of those with **no** repeated digit is $10 \cdot 9 \cdots 4 = 10!/3!$, so the probability that a random 7-digit integer has no repeated digit is $(10!/3!)/(10^7)$. The probability asked in the problem is that of the complement, i.e.,

$$\boxed{1 - \frac{10!/3!}{10^7}}$$

(b) What is the probability that a random 7-digit integer contains the digit 0 exactly 3 times?

Solution. This is a success/failure trial situation, with the digits corresponding to trials, success corresponding to a 0 digit (which has probability $p = 1/10$) and failure corresponding to a non-0 digit. The probability asked for is that of getting 3 successes in 7 trials with $p = 1/10 = 0.1$, which is given by the binomial distribution:

$$\boxed{\binom{7}{3} 0.1^3 0.9^4 \; \left(= \frac{\binom{7}{3} 9^4}{10^7}\right)}$$

(c) What is the probability that in a random 7-digit integer all digits are distinct and occur in increasing order (e.g., 0134789). (The answer should be a simple expression in "raw" form, not a complicated sum.)

Solution. There are $\binom{10}{7}$ ways to choose the 7 digits; once these digits have been chosen, there is only one way to put them in increasing order. Thus, the probability is

$$\boxed{\frac{\binom{10}{7}}{10^7}}$$