

Math 408, Spring 2008
Midterm Exam 2 Solutions

1. (NO CALCULATORS FOR THIS PROBLEM)

(a) Consider the following statements:

- (1) If $\text{Cov}(X, Y) = 0$, then X and Y are independent.
- (2) If X and Y are independent, then $\text{Cov}(X, Y) = 0$.

Circle one of the following:

- (i) Only (1) is correct.
- (ii) Only (2) is correct.
- (iii) (1) and (2) are both correct.
- (iv) Neither (1) nor (2) are correct.

Solution. (ii), i.e., (2) is correct, but not (1). (Note that, while independence implies that the covariance is zero, the converse is not true. In fact, a homework problem (4.2-7) gave an example of two random variables that had 0 covariance, but were not independent.)

(b) Suppose that X and Y are independent random variables with $\text{Var}(X) = 1$, $\text{Var}(Y) = 2$. Find $\text{Var}(1 - 2X + 3Y)$.

Solution. (Except for a minor numerical change, this was a quiz problem.)

$$\text{Var}(1 - 2X + 3Y) = 0 + (-2)^2 \text{Var}(X) + 3^2 \text{Var}(Y) = 4 \cdot 1 + 9 \cdot 2 = \boxed{22}.$$

(c) Suppose X and Y are random variables such that $\text{Var}(X + Y) = 9$ and $\text{Var}(X - Y) = 1$. Find $\text{Cov}(X, Y)$.

Solution. We have

$$\begin{aligned}\text{Var}(X + Y) &= \text{Var}(X) + \text{Var}(Y) + 2 \text{Cov}(X, Y), \\ \text{Var}(X - Y) &= \text{Var}(X) + \text{Var}(-Y) + 2 \text{Cov}(X, -Y) \\ &= \text{Var}(X) + \text{Var}(Y) - 2 \text{Cov}(X, Y),\end{aligned}$$

(since $\text{Var}(-Y) = \text{Var}(Y)$ and $\text{Cov}(X, -Y) = -\text{Cov}(X, Y)$). Subtracting these two equations and substituting the given values for $\text{Var}(X + Y)$ and $\text{Var}(X - Y)$, we get $9 - 1 = 4 \text{Cov}(X, Y)$, so $\text{Cov}(X, Y) = \boxed{2}$.

2. (NO CALCULATORS FOR THIS PROBLEM) Let X be a random variable with normal distribution $N(10, 4)$, and let $Y = e^X$.

- (a) Find the p.d.f. of X . (The answer should be an explicit elementary function of x , not an expression involving Φ .)

Solution. The density of a general normal distribution with parameters μ and σ is $f(x) = (1/\sqrt{2\pi}\sigma)e^{-(x-\mu)^2/(2\sigma^2)}$. Here $\mu = 10$, $\sigma = 2$, so

$$f(x) = \frac{1}{2\sqrt{2\pi}}e^{-\frac{1}{8}(x-10)^2}, \quad -\infty < x < \infty.$$

- (b) Find the p.d.f. of Y . (Again, the answer should be an explicit elementary function of y .)

Solution. We use the distribution function technique to compute first the c.d.f. $G(y)$ and then the p.d.f. $g(y)$ of Y . Since $Y = e^X$ and X ranges over the entire real axis, the possible values for Y are $0 < y < \infty$. For such y we have

$$G(y) = P(Y \leq y) = P(e^X \leq y) = P(X \leq \ln y) = F(\ln y).$$

Differentiating and using the formula for $f(x)$ from part (a), we get

$$\begin{aligned} g(y) &= G'(y) = F'(\ln y) \frac{1}{y} = \frac{f(\ln y)}{y} \\ &= \frac{1}{2y\sqrt{2\pi}}e^{-\frac{1}{8}(\ln y - 10)^2}, \quad 0 < y < \infty. \end{aligned}$$

3. (NO CALCULATORS FOR THIS PROBLEM) Claim amounts on a home insurance policy have density function

$$f(x) = \begin{cases} 3x^{-4} & \text{for } x > 1, \\ 0 & \text{otherwise.} \end{cases}$$

Suppose two such claims are made, and assume the claim amounts are independent.

- (a) Find the p.d.f. (density) of the *larger* of the two claims.

Solution. Let X_1 and X_2 denote the two claims, and let $X^* = \max(X_1, X_2)$ be the larger (maximum) of these claims. To find the p.d.f. $g(x)$ of X^* , we first compute the c.d.f. $G(x) = P(X^* \leq x)$, using the “maximum trick”: For $x \geq 1$,

$$\begin{aligned} G(x) &= P(X^* \leq x) = P(\max(X_1, X_2) \leq x) \\ &= P(X_1 \leq x, X_2 \leq x) = P(X_1 \leq x)P(X_2 \leq x) = F(x)^2. \end{aligned}$$

Now, $f(x) = 3x^{-4}$ for $x \geq 1$, so $F(x) = \int_1^x 3t^{-4} dt = 1 - x^{-3}$, and therefore

$$\begin{aligned} G(x) &= (1 - x^{-3})^2, \\ g(x) &= G'(x) = 2(1 - x^{-3})(-(-3)x^{-4}) \\ &= \boxed{6x^{-4} - 6x^{-7} \quad (x \geq 1)}. \end{aligned}$$

- (b) Find the probability that the amount of the *second* claim is at least twice that of the *first* claim.

Solution. Let X_1 and X_2 denote, respectively, the first and second claims. Then we need to compute $P(X_2 \geq 2X_1)$. By the independence of X_1 and X_2 and the given density for an individual claim, the joint density is

$$f(x_1, x_2) = (3x_1^{-4})(3x_2^{-4}) = 9x_1^{-4}x_2^{-4}, \quad 1 < x_1, x_2 < \infty.$$

Hence

$$\begin{aligned} P(X_2 > 2X_1) &= \int_{x_1=1}^{\infty} \int_{x_2=2x_1}^{\infty} 9x_1^{-4}x_2^{-4} dx_2 dx_1 = \int_{x_1=1}^{\infty} 9x_1^{-4} \left[\frac{x_2^{-3}}{-3} \right]_{x_2=2x_1}^{\infty} dx_1 \\ &= \int_{x_1=1}^{\infty} 3x_1^{-4}(2x_1)^{-3} dx = \frac{3}{8} \int_{x_1=1}^{\infty} x_1^{-7} dx = \boxed{\frac{1}{16}}. \end{aligned}$$

4. (NO CALCULATORS FOR THIS PROBLEM) Let X and Y be random variables with joint density

$$f(x, y) = \frac{3}{2}, \quad 0 \leq x \leq 1, x^2 \leq y \leq 1.$$

- (a) Find the marginal density $f_Y(y)$ of Y . Be sure to specify the range.

Solution. The general formula is $f_Y(y) = \int_*^* f(x, y) dx$ with appropriate integration limits. A sketch of the region shows that the correct limits are from $x = 0$ to $x = \sqrt{y}$. Thus,

$$f_Y(y) = \int_{x=0}^{\sqrt{y}} \frac{3}{2} dx = \boxed{\frac{3}{2}y^{1/2}, \quad 0 \leq y \leq 1}.$$

- (b) Find $E(X|1/2)$, the conditional expectation of X given that $Y = 1/2$.

Solution. The conditional density of X given $Y = 1/2$ is

$$h(x|1/2) = \frac{f(x, 1/2)}{f_Y(1/2)} = \frac{3/2}{(3/2)\sqrt{1/2}} = \boxed{\sqrt{2}, \quad 0 < x < \sqrt{1/2}}.$$

Thus

$$E(X|1/2) = \int xh(x|1/2)dx = \int_0^{\sqrt{1/2}} x\sqrt{2}dx = \sqrt{2}\frac{(1/\sqrt{2})^2}{2} = \boxed{\frac{1}{2\sqrt{2}}}.$$

5. (NO CALCULATORS FOR THIS PROBLEM) Suppose that Y is uniformly distributed on the interval $[0, 4]$ and that, given $Y = y$, X is uniformly distributed on the interval $[y, y + 2]$.

- (a) Determine the joint density $f(x, y)$. Be sure to specify the range.

Solution. Since Y is uniformly distributed on $[0, 4]$, we have $f_Y(y) = 1/4$, $0 \leq y \leq 2$. Similarly, since, given $Y = y$, X is uniformly distributed on $[y, y + 2]$, the conditional density of X given $Y = y$ is $1/2$ on the interval $[y, y + 2]$; i.e., $g(x|y) = 1/2$, $y \leq x \leq y + 2$ for $0 \leq y \leq 4$. Thus

$$f(x, y) = f_Y(y)g(x|y) = \frac{1}{4} \cdot \frac{1}{2} = \boxed{\frac{1}{8}, \quad 0 < y < 4, y < x < y + 2}$$

- (b) Find the marginal density $f_X(x)$ of X . Be sure to specify the range.

Solution. [This is Problem 4.3-14 in Hogg/Tanis, from HW Assignment 10. The key is to split up the x -range into three parts, and to calculate $f_X(x)$ separately for each of the three parts.] The general formula is $f_X(x) = \int_*^* f(x, y)dy$, with appropriate integration limits. From part (a) we have $f(x, y) = 1/8$, and from a sketch we see that the correct integration limits are $0 \leq y \leq x$ for $0 \leq x \leq 2$, $x - 2 \leq y \leq x$ for $2 \leq x \leq 4$, $x - 2 \leq y \leq 4$ for $4 \leq x \leq 6$. Hence,

$$f_X(x) = \begin{cases} \int_0^x \frac{1}{8} dy = \frac{x}{8} & \text{if } 0 \leq x \leq 2, \\ \int_{x-2}^x \frac{1}{8} dy = \frac{1}{4} & \text{if } 2 \leq x \leq 4, \\ \int_{x-2}^4 \frac{1}{8} dy = \frac{6-x}{8} & \text{if } 4 \leq x \leq 6. \end{cases}$$

6. Assume that the amount of fluid content contained in a can of soda has normal distribution with mean 12.2 ounces and standard deviation 0.1 ounces.

- (a) Find the probability that a single can of soda contains less than 12 ounces.

Solution. We need to compute $P(X < 12)$ where X is normal $N(12.2, 0.1^2)$. Standardizing, we get

$$P(X < 12) = P\left(\frac{X - 12.2}{0.1} < \frac{12 - 12.2}{0.1}\right) = P(Z < -2) = 1 - \Phi(2) = \boxed{0.0228}.$$

- (b) Suppose that 50 cans are measured, and that the 50 measurements are averaged. Assuming independence, determine a value c such that, with probability 0.0228, the *average* of these 50 measurements is less than c .

Solution. We need to determine c such that $P(\bar{X} < c) = 0.9772$, where \bar{X} is the sample mean of the 50 r.v.'s X_1, \dots, X_{50} . By the CLT, \bar{X} is approximately normal with mean $\mu = 12.2$ and standard deviation $0.1/\sqrt{50} = 0.014\dots$. Standardizing, we have

$$\begin{aligned} P(\bar{X} < c) &= P\left(\frac{\bar{X} - 12.2}{0.014} < \frac{c - 12.2}{0.014}\right) \\ &= P\left(Z < \frac{c - 12.2}{0.014}\right) = \Phi\left(\frac{c - 12.2}{0.014}\right) \end{aligned}$$

Setting this equal to 0.0228, we find that $(c - 12.2)/0.014 = -2$, so $c = 12.2 - 0.028 = \boxed{12.17}$.