Math 408, Actuarial Statistics I, Spring 2008

Solutions to combinatorial problems

Word counting problems

1. Find the number of possible 10 character passwords under the following restrictions: (Note there are 26 letters in the alphabet.)

   (a) All characters must be lower case letters.

   Solution: 26^{10}

   (b) All characters must be lower case letters and distinct.

   Solution: $26 \cdot 25 \cdot 24 \cdots 17 (= 26!/16!)$

   (c) Letters and digits must alternate and be distinct (as in "1w2x9c4u5s" or "a1b2c3d4e5").

   Solution: $2 \cdot (26 \cdot 25 \cdot 24 \cdot 23 \cdot 22) \cdot (10 \cdot 9 \cdot 8 \cdot 7 \cdot 6)$: 2 counts the two possible patters LDLDLDLDLD and DLDLDLDLDL, where L and D denote letters and digits; $26 \cdots 22$ is the number of ways to fill the L slots with distinct letters, and $10 \cdots 6$ is the number of ways to fill the D slots with distinct digits chosen from $0, 1, \dots, 9$. (This assumes that only lower case letters are allowed.)

   (d) All characters must be lower case, distinct, and *in alphabetical order*. (e.g., "abfghikmno" is allowed, but not "bafghikmno").

   Solution: $\binom{26}{10}$, since such words correspond exactly to unordered samples of size 10 without replacement from the 26 letters. (Given such a sample, there is exactly one way to place the letters in the sample in alphabetical order to form a word of the required type. Conversely, any word of this type induces a sample of the above form.)

   (e) The word can only contain the upper case letters A and B.

   Solution: 2^{10}: There are 2 choices for each of the 10 slots for the letters, giving $\underbrace{2 \cdot 2 \cdots 2}_{10} = 2^{10}$ choices altogether.

   (f) The word can only contain the upper case letters A and B, and must contain each of these letters.

   Solution: $2^{10} - 2$: Subtract 2 from the previous count to exclude the words consisting of all A's or all B's.

   (g) The word can only contain the upper case letters A and B, and must contain an equal number of each.

   Solution: $\binom{10}{5}$: Choose 5 slots out of the 10 available to fill with A's ($\binom{10}{5}$ ways to do this), and put B's into the remaining slots ($\binom{5}{5} = 1$ ways).

2. Find the number of different words that can be formed by rearranging the letters in the following words: (Include the given word in the count.)

   (a) NORMAL

   Solution: 6!: Since the letters in NORMAL are all distinct, the rearrangements are just the permutations of the 6 letters.

(b) HHTTTT

Solution: $\binom{6}{2}$: Start with 6 blank slots for the letters. Then fill 2 of these with the letter H, and the remaining 4 with the letter T. There are $\binom{6}{2}$ ways to choose the slots for the H, and once the H's have been placed, there are $\binom{4}{4} = 1$ ways to place the 4 T's in the remaining 4 slots. Thus, the total count is $\binom{6}{2}$.

(c) ILLINI

Solution: $\binom{6}{3}\binom{3}{2} = \frac{6!}{3!2!1!}$: As before start with 6 blank slots for the letters. Then fill 3 of these with the letter I; there are $\binom{6}{3}$ ways to do this. Of the 3 remaining slots, fill 2 with the letter L; there are $\binom{3}{2}$ ways for this step. Finally, fill the one remaining slot with N; there is only one way to do this step. Hence the total count is $\binom{6}{3}\binom{3}{2}$. The second formula, $6!/(3!2!1!)$, is obtained by expressing the binomial coefficients in terms of factorials and simplifying.

(d) MISSISSIPPI

Solution: $\binom{11}{4}\binom{7}{4}\binom{3}{2} = \frac{11!}{4!4!2!1!}$: This follows by the same argument as before, using the 4 S's, 4 I's, 2 P's, and 1 M that make up the given word.

3. How many ways are there to seat 10 people, consisting of 5 couples, in a row of seats (10 seats wide) if

(a) the seats are assigned at random?

Solution: $10!$: The possible seating orders are just permutations of $1, 2, \dots, 10$.

(b) all couples are to get adjacent seats?

Solution: $10 \cdot 8 \cdot 6 \cdot 4 \cdot 2$: Fill 10 slots (seats) left to right keeping in mind the restriction that couples have to sit together. There are 10 choices for the first slot (no restriction), 1 choice for the second (since the first person's mate has to be seated there), 8 choices for the third slot (since 2 people have now been seated, leaving 8), 1 choice for the fourth slot (since this seat has to go to the mate of # 3), etc.

4. Assume the final exams are given over a period of 6 days, with 3 slots per day, so that there are a total of 18 final exam slots. If you have 5 classes, each with a final exam, what is the probability that your 5 finals fall onto different days, assuming that the 5 classes all have different final exam slots?

Solution: Encode the final exam slots for the 5 classes as a 5-letter "word", with each character being one of the 18 available final exam periods. By assumption, the periods are all different, so the total number of such "words" is $18 \cdot 17 \cdot 16 \cdot 15 \cdot 14$. To count the words that satisfy the additional requirement that the final periods all fall onto different days, imagine filling the slots left to right and counting the number of possibilities at each stage. This gives $18 \cdot 15 \cdot 12 \cdot 9 \cdot 6$ as count. The probability sought is the quotient between these two numbers, $(18 \cdot 15 \cdot 12 \cdot 9 \cdot 6)/(18 \cdot 17 \cdots 14)(= 0.17 \dots)$.

Birthday type problems

1. What is the probability that in a group of n no two people share a common birthday? Work out the probabilities numerically for $n = 20, 30, 40, 50, 60$. (Assume there are 365 days in a year.)

Solution: The birthdays of the n people can be encoded as an n-letter "word", with each letter being one of the 365 possible birthdays in the year. The number of such words, with or without the restriction that the birthdays be distinct, can be counted in the usual way, by constructing the words left-to-right by filling "slots" and counting the number of choices at each slot. Without restriction, we get a total count $\underbrace{365 \cdot 365 \cdots 365}_{n} = 365^n$; with the restriction that the birthdays be distinct we get $365 \cdot 364 \cdots (365-n+1)$.

Thus, the probability that all n birthdays are distinct is $365 \cdot 364 \cdots (365-n+1)/365^n$. The numerical values for $n = 20, 30, 40, 50, 60$ are 0.5885, 0.2936, 0.1087, 0.0296, 0.0058.

2. Six people get into an elevator at the ground floor of a hotel which has 10 upper floors. Assuming each person gets off at a randomly chosen floor, what is the probability that no two people get off at the same floor?

   Solution: $10 \cdot 9 \cdots 5/10^6 (= 0.15\dots)$. This is just like the birthday problem, with the 10 floors corresponding to the 365 birthdays.

3. Suppose you record the birthdays of a large group of people, one at a time (just like in the birthday simulation posted on the course webpage) until you have found a match, i.e., a birthday that has already been recorded.

   (a) What is the probability that it takes *more than* 20 people for this to occur?

   Solution: It takes more than 20 people if and only if among the first 20 that are asked there is no matching birthday. The latter has probability $365 \cdot 364 \cdots 346/365^{20}$, by the standard version of the birthday problem.

   (b) What is the probability that it takes *exactly* 20 people for this to occur?

   Solution: This requires (a) the first 19 birthdays to be distinct and (b) the 20th birthday to be equal to one of the first 19. There are $365 \cdot 364 \cdots 366$ ways to fill 19 slots with distinct birthdays and 19 ways to fill the 20th slot with one of the birthdays in the other 19 slots. Since the total (unrestricted) number of ways to fill 20 slots with birthdays is 365^{20}, the above event has probability $365 \cdot 364 \cdots 345 \cdot 19/365^{20}$.

4. Suppose you record the birthdays of a large group of people, one at a time, until you have found a person whose birthday *matches your own birthday*. What is the probability that it takes exactly 20 people for this to occur?

   Solution: Despite appearances, this is quite different from the standard birthday problem as we are now looking for a birthday matching a *given* birthday (namely, yours). There is a 1/365 chance of a randomly chosen birthday to match the given birthday, so the problem can be modeled by repeated success/failure trials, with success meaning a birthday matching the given birthday and occurring with probability $p = 1/365$. In terms of this model the probability sought is the probability that the first success occurs at the 20th trial. This probability is given by the geometric distribution $(1-p)^{n-1}p$ with $n = 20$ and $p = 1/365$, i.e., it is $(364/365)^{19}(1/365) (= 0.0026\dots)$.

Urn/ball type problems

1. **The classical urn/ball problem.** An urn (box) contains 30 balls, of which 10 are red and the other 20 blue. Suppose you take out 8 balls from this urn, without replacement. What is the probability that among the 8 balls in this sample exactly 3 are red and 5 are blue?

   Solution: The total number of ways to take out 8 balls is the number of unordered samples of size 8, without replacement, from 30, i.e., $\binom{30}{8}$. This is the denominator in the probability we need to compute. For the numerator, we need to count those samples that consist of exactly 3 red and 5 blue balls. These samples can be obtained in two stages, by first picking 3 out of the 10 red balls ($\binom{10}{3}$ ways to do this), and then picking 5 out of the 20 blue balls ($\binom{20}{5}$ ways to do this). The total number of such samples is therefore $\binom{10}{3}\binom{20}{5}$, and the probability sought is $\binom{10}{3}\binom{20}{5}/\binom{30}{8} (= 0.31\dots)$.

2. **The committee problem.** Assume a committee of 10 has to be selected from a group of 100 people, of which 40 are men and 60 are women.

   (a) How many ways are there to choose such a committee?

   Solution: $\binom{100}{10}$

   (b) How many ways are there to choose the committee so that exactly half of the members are men?

   Solution: $\binom{40}{5}\binom{60}{5}$

   (c) What is the probability that a randomly selected committee of 10 consists of exactly 5 men and 5 women?

   Solution: $\binom{40}{5}\binom{60}{5}/\binom{100}{10}$: This follows by the same argument as for the urn/ball problem.

3. **The lottery problem.** In a state lottery a player has to choose 6 (distinct) numbers out of 54 numbers. At each drawing, 6 numbers are drawn at random, without replacement, from these 54 numbers. You win a first prize if all 6 drawn numbers agree with the ones you picked, a second prize if exactly 5 of the drawn numbers agree, etc. Compute the probabilities of winning a first prize, a second prize, etc.

   Solution: This can be viewed as an urn/ball type problem, with a total of 54 balls (corresponding to the lottery numbers), red balls corresponding to the 6 lottery numbers you picked, and blue balls corresponding to the remaining 48 numbers. A lottery drawing can be thought of as an unordered sample of size 6 without replacement, and the prize depends on how many "red" balls (i.e., your numbers) are among those drawn. For first prize, the probability is $\binom{6}{6}\binom{48}{0}/\binom{54}{6}$, for second prize, $\binom{6}{5}\binom{48}{1}/\binom{54}{6}$, etc.

Additional examples/problems from Hogg/Tanis

1. Problem 1.3-1

   Solution: 8^4.

2. Problem 1.3-3

   Solution: (a) $26^2 \cdot 10^4$; (b) $26^3 \cdot 10^3$

3. Problem 1.3-5

   Solution: (a) $4!$; (b) 4^4

4. Problem 1.3-7

   Solution: Here a lottery drawing can be thought of as an ordered sample of size 4, with replacement, from the 10 digits $0, 1, \ldots, 9$. The total number of these is 10^4, which is the denominator in the probabilities sought.
   (a) If you choose $6, 7, 8, 9$, there are $4! = 24$ permutations in which these numbers can occur, so the probability of winning is $24/10^4$.
   (b) If you choose $6, 7, 8, 8$, there are $\binom{4}{2}\binom{2}{1} = 12$ ways to rearrange these 4 digits, by the same argument as that used above for counting rearrangements of the letters of ILLINI. Thus, the probability becomes $12/10^4$.
   (c) For $7, 7, 8, 8$, there are $\binom{4}{2} = 6$ rearrangements, so the probability is $6/10^4$.
   (c) For $7, 8, 8, 8$, there are $\binom{4}{3} = 4$ rearrangements, so the probability is $4/10^4$.

5. Problem 1.3-11

Solution: This boils down to a careful case-by-case analysis, keeping in mind that the series ends as soon as one of the teams wins four games. Thus, for example, a pattern like ANNNNA is not possible, since the series would be over in 5 games. Here are the cases:
(a) Four game series: The only possible patters are AAAA, NNNN, so there are 2 ways for the series to end in four games.
(b) Five game series: Assume first that A wins the series. Then the 5th game has to go to A, and out of the first 4 games A wins exactly 3. There are $\binom{4}{3}$ such patterns resulting in A winning in 5 games. (This is just like counting H/T sequences with 3 H's and 1 T.) Doubling that number to account for the cases when N wins in 5 games, we get $2 \cdot \binom{4}{3} = 8$ ways for the series to last 5 games.
(c) Six game series: $2 \cdot \binom{5}{3} = 20$ ways, by the same argument.
(d) Seven game series: $2 \cdot \binom{6}{3} = 40$ ways, by the same argument.