Combinatorial Probabilities

Key concepts

- **Permutation:** arrangement in some order.
- **Ordered versus unordered samples:** In ordered samples, the order of the elements in the sample matters; e.g., digits in a phone number, or the letters in a word. In unordered samples the order of the elements is irrelevant; e.g., elements in a subset, or lottery numbers.
- **Samples with replacement versus samples without replacement:** In the first case, repetition of the same element is allowed (e.g., numbers in a license plate); in the second, repetition not allowed (as in a lottery drawing—once a number has been drawn, it cannot be drawn again).

Formulas

- Number of **permutations** of n objects: $n!$
- Number of **ordered** samples of size r, **with** replacement, from n objects: n^r
- Number of **ordered** samples of size r, **without** replacement, from n objects:

$$n(n-1)\cdots(n-r+1) = \frac{n!}{(n-r)!} = {}_nP_r.$$

- Number of **unordered** samples of size r, **without** replacement, from a set of n objects (= number of subsets of size r from a set of n elements) (**combinations**):

$$\binom{n}{r} = \frac{{}_nP_r}{r!} = \frac{n!}{r!(n-r)!} = \frac{n(n-1)\dots(n-r+1)}{r!}.$$

  (See back of page for properties of these binomial coefficients.)

- Number of **subsets** of a set of n elements: 2^n

Binomial coefficients

- **Definition:** For $n = 1, 2, \ldots$ and $k = 0, 1, \ldots, n$, $\binom{n}{k} = \dfrac{n!}{k!(n-k)!}$.
  (Note that, by definition, $0! = 1$.)
- **Alternate notations:** ${}_nC_k$ or $C(n, k)$
- **Alternate definition:** $\binom{n}{k} = \dfrac{n(n-1)\ldots(n-k+1)}{k!}$.
  (This version is convenient for hand-calculating binomial coefficients.)
- **Symmetry property:** $\binom{n}{k} = \binom{n}{n-k}$
- **Special cases:** $\binom{n}{0} = \binom{n}{n} = 1, \quad \binom{n}{1} = \binom{n}{n-1} = n$
- **Binomial Theorem:** $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$
- **Binomial Theorem, special case:** $\sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} = 1$
- **Combinatorial Interpretations:** $\binom{n}{k}$ represents
  1. the number of ways to select k objects out of n given objects (in the sense of unordered samples without replacement);
  2. the number of k-element subsets of an n-element set;
  3. the number of n-letter HT sequences with exactly k H's and $n-k$ T's.
- **Binomial distribution:** Given a positive integer n and a number p with $0 < p < 1$, the binomial distribution $b(n, p)$ is the distribution with density (p.m.f.) $f(k) = \binom{n}{k} p^k (1-p)^{n-k}$, for $k = 0, 1, \ldots, n$.