Math 370/408, Spring 2008
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Actuarial Exam Practice Problem Set 5
Solutions

About this problem set: These are problems from Course 1/P actuarial exams that I have collected over the years, grouped by topic, and ordered by difficulty. All of the exams have appeared on the SOA website `http://www.soa.org/` at some point in the past, though most of them are no longer there. The exams are copy-righted by the SOA.

Note on the topics covered: This problem set covers problems on the normal distribution and the Central Limit Theorem (Chapter 5 of Hogg/Tanis).

Note on the ordering of the problems: The problems are loosely grouped by topic, and very roughly ordered by difficulty (though that ordering is very subjective). The last group of problems (those with triple digit numbers) are harder, either because they are conceptually more difficult, or simply because they require lengthy computations and may take two or three times as long to solve than an average problem.

Answers and solutions: I will post an answer key and hints/solutions on the course webpage, `www.math.uiuc.edu/~hildebr/370`.

1. [5-1]

   Claims filed under auto insurance policies follow a normal distribution with mean $19,400$ and standard deviation $5,000$. What is the probability that the average of 25 randomly selected claims exceeds $20,000$?

   (A) 0.01 (B) 0.15 (C) 0.27 (D) 0.33 (E) 0.45

   Answer: C: 0.27

   Solution: The probability to compute is $P(\overline{X} \geq 20,000)$, where $\overline{X}$ is the mean of a random sample of size 25 from a normal distribution with mean $19,400$ and with standard deviation $5,000$. By the CLT (or since the sum of independent normals is again normal), $\overline{X}$ has normal distribution with the same mean and with standard deviation $5,000/\sqrt{25} = 1,000$. After standardizing, we get

$$\begin{aligned} P(\overline{X} \geq 20,000) &= P\left(\frac{\overline{X} - 19,400}{1000} \geq \frac{20,000 - 19,4000}{1000}\right) \\ &= P(Z \geq 0.6) = 1 - \Phi(0.6) = 0.275. \end{aligned}$$

2. [5-2]

   The total claim amount for a health insurance policy follows a distribution with density function

$$f(x) = \frac{1}{1000} e^{-x/1000}, \quad \text{for } x > 0.$$

   The premium for the policy is set at 100 over the expected total claim amount. If 100 policies are sold, what is the approximate probability that the insurance company will have claims exceeding the premiums collected?

   (A) 0.001 (B) 0.159 (C) 0.333 (D) 0.407 (E) 0.460

   Answer: B: 0.159

   Solution: The given density can be recognized as an exponential density with mean 1000 and standard deviation 1000. The total of 100 independent claims with this density is a sum S of 100 independent r.v., each having the given density. By the CLT, the standardized version of S, $Z = (S - 100,000)/10,000$, has approximately normal distribution. The probability to compute is therefore $P(S \geq 110,000) = P(Z \geq 1)$, which is $1-\Phi(1) = 0.159$.

3. [5-3]

   In an analysis of healthcare data, ages have been rounded to the nearest multiple of 5 years. The difference between the true age and the rounded age is assumed to be uniformly distributed on the interval from -2.5 years to 2.5 years. The healthcare data are based on a random sample of 48 people. What is the approximate probability that the mean of the rounded ages is within 0.25 years of the mean of the true ages?

   (A) 0.14 (B) 0.38 (C) 0.57 (D) 0.77 (E) 0.88

   Answer: D: 0.77

   Solution: We need to compute $P(|\overline{X}| \leq 0.25)$, where $\overline{X}$ is the mean of a random sample of size 48 from a uniform distribution on $[-2.5, 2.5]$. Obviously this distribution has mean 0, and an easy computation (or using the formula $\sigma^2 = (b-a)^2/12$ for the variance of a uniform distribution on an interval $[a, b]$) shows that the standard deviation is 1.44. By the CLT, $\overline{X}$ has approximately normal distribution with mean 0 and standard deviation $1.44/\sqrt{48} = 0.2078$, so

$$P(|\overline{X}| \le 0.25) = P\left(\left|\frac{\overline{X}-0}{0.2078}\right| \le \frac{0.25}{0.2078}\right)$$
$$\approx P(|Z| \le 1.2) = \Phi(1.2) - \Phi(-1.2) = 0.769.$$

4. [5-4]

An insurance company issues 1250 vision care insurance policies. The number of claims filed by a policyholder under a vision care insurance policy during one year is a Poisson random variable with mean 2. Assume the numbers of claims filed by distinct policyholders are independent of one another. What is the approximate probability that there is a total of between 2450 and 2600 claims during a one-year period?

(A) 0.68 (B) 0.82 (C) 0.87 (D) 0.95 (E) 1.00

Answer: B: 0.82

Solution: The total number of claims is a sum S of 1250 independent random variables, each having Poisson distribution with mean $\mu = 2$. Since the variance of a Poisson distribution is equal to its mean, the standard deviation of these r.v.'s is $\sigma = \sqrt{2}$. By the CLT, it follows that S is approximately normal with mean $1250 \cdot 2 = 2500$ and standard deviation $\sqrt{1250} \cdot \sqrt{2} = 50$. Hence the probability to compute is

$$P(2450 \le S \le 2600) = P\left(\frac{2450-2500}{50} \le \frac{S-2500}{50} \le \frac{2600-2500}{50}\right)$$
$$\approx P(-1 \le Z \le 2) = \Phi(2) - \Phi(-1) = 0.82.$$

5. [5-5]

A charity receives 2025 contributions. Contributions are assumed to be independent and identically distributed with mean 3125 and standard deviation 250. Calculate the approximate 90th percentile for the distribution of the total contributions received.

(A) $6,328,000$ (B) $6,338,000$ (C) $6,343,000$ (D) $6,784,000$ (E) $6,977,000$

Answer: C: 6,343,000

Solution: If S denotes the total (i.e., sum) of all contributions received, we need to find x such that, approximately, $P(S \le x) = 0.9$. Now S is the sum of 2025 independent r.v.'s, each with mean $\mu = 3125$ and standard deviation 250. Hence, by the CLT, S is approximately normal with mean $1250 \cdot 2025 = 6328125$ and standard deviation $250 \cdot \sqrt{2025} = 11250$, and so

$$P(S \le x) = P\left(\frac{S-6328125}{11250} \le \frac{x-6328125}{11250}\right)$$
$$\approx P\left(Z \le \frac{x-6328125}{11250}\right) = \Phi\left(\frac{x-6328125}{11250}\right).$$

Setting this equal to 0.9, we get from the normal table $(x - 6328125)/11250 = 1.28$, and so $x = 6328125 + 11250 \cdot 1.28 = 6342525$.

6. [5-6]

A company manufactures a brand of light bulb with a lifetime in months that is normally distributed with mean 3 and variance 1. A consumer buys a number of these bulbs with the intention of replacing them successively as they burn out. The light bulbs have independent lifetimes. What is the smallest number of bulbs to be purchased so that the succession of light bulbs produces light for at least 40 months with probability at least 0.9772 ?

(A) 14 (B) 16 (C) 20 (D) 40 (E) 55

Answer: B: 16

Solution: This problem is different from the standard CLT fare in that the unknown here is the number n of r.v.'s needed to achieve a given probability. The set-up, however, is the same: Let S be the sum of n r.v.'s representing the successive lifetimes of the bulbs. We are given that these lifetimes are independent and have mean 3 and variance (and standard deviation) 1. Thus, by the CLT (assuming n is reasonably large), S has approximately normal distribution with mean $3n$ and standard deviation $\sqrt{n}$. We need to choose n so that $P(S \geq 40)$ is (approximately) 0.9772. Converting to standard units, we have

$$\begin{aligned} P(S \geq 40) &= P\left(\frac{S-3n}{\sqrt{n}} \geq \frac{40-3n}{\sqrt{n}}\right) \\ &\approx P\left(Z \geq \frac{40-3n}{\sqrt{n}}\right) = 1 - \Phi\left(\frac{40-3n}{\sqrt{n}}\right) = \Phi\left(-\frac{40-3n}{\sqrt{n}}\right). \end{aligned}$$

Setting this equal to 0.9772, we get from the normal table $-(40-3n)/\sqrt{n} = 2$, or $3n - 2\sqrt{n} - 40 = 0$. This is a quadratic equation in $\sqrt{n}$. Solving, we find $\sqrt{n} = 4$. (The other solution is negative and thus can be discarded.) Hence $n = 4^2 = 16$.

7. [5-7]

Two instruments are used to measure the height, h, of a tower. The error made by the less accurate instrument is normally distributed with mean 0 and standard deviation $0.0056h$. The error made by the more accurate instrument is normally distributed with mean 0 and standard deviation $0.0044h$. Assuming the two measurements are independent random variables, what is the probability that their average value is within $0.005h$ of the height of the tower?

(A) 0.38 (B) 0.47 (C) 0.68 (D) 0.84 (E) 0.90

Answer: D: 0.84

Solution: If X and Y denote the two measurements, then their average is $A = (1/2)X + (1/2)Y$. We need to compute $P(|A| \leq 0.005h)$. Since A is a linear combination of independent normals, it is normal, and the mean and variance of A can be computed as follows:

$$\begin{aligned} E(A) &= E((1/2)X + (1/2)Y) = 0, \\ \mathrm{Var}(A) &= \mathrm{Var}((1/2)X + (1/2)Y) \\ &= \frac{1}{4}\mathrm{Var}(X) + \frac{1}{4}\mathrm{Var}(Y)) = \frac{1}{4}(0.0056h^2 + 0.0044h^2) = 0.00001268h^2. \end{aligned}$$

Thus, after standardizing A by dividing by $\sqrt{0.00001268^2h^2} = 0.00356h$, the probability to compute becomes

$$\begin{aligned} P(|A| \leq 0.005h) &= P\left(\left|\frac{A}{0.00356h}\right| \leq \frac{0.005h}{0.00356h}\right) \\ &= P(|Z| \leq 1.4) = \Phi(1.4) - \Phi(-1.4) = 0.838. \end{aligned}$$

8. [5-51]

   Let X and Y be the number of hours that a randomly selected person watches movies and sporting events, respectively, during a three-month period. The following information is known about X and Y:

$$E(X) = 50, \quad E(Y) = 20, \quad \text{Var}(X) = 50, \quad \text{Var}(Y) = 30, \quad \text{Cov}(X, Y) = 10.$$

   One hundred people are randomly selected and observed for these three months. Let T be the total number of hours that these one hundred people watch movies or sporting events during this three-month period. Approximate the value of $P(T < 7100)$.

   (A) 0.62 (B) 0.84 (C) 0.87 (D) 0.92 (E) 0.97

   Answer: B: 0.84

   Solution: The total time that a *single* person spends watching is a random variable with the distribution of $X + Y$ and mean and variance given by

$$\mu = E(X + Y) = E(X) + E(Y) = 50 + 20 = 70,$$
$$\sigma^2 = \text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y) + 2\,\text{Cov}(X, Y) = 50 + 30 + 2 \cdot 10 = 100.$$

   The total time, T, spent by 100 such people is a sum of 100 independent random variables, each of the type $X + Y$. By the CLT, T has approximately normal distribution with mean $100\mu = 100 \cdot 70 = 7000$ and standard deviation $\sqrt{100}\sigma = \sqrt{100} \cdot \sqrt{100} = 100$. Hence

$$P(T < 7100) = P\left(\frac{T - 7000}{100} < \frac{7100 - 7000}{100}\right) \approx P(Z < 1) = \Phi(1) = 0.84.$$

9. [5-52]

   A city has just added 100 new female recruits to its police force. The city will provide a pension to each new hire who remains with the force until retirement. In addition, if the new hire is married at the time of her retirement, a second pension will be provided for her husband. A consulting actuary makes the following assumptions:

   (i) Each new recruit has a 0.4 probability of remaining with the police force until retirement.

   (ii) Given that a new recruit reaches retirement with the police force, the probability that she is not married at the time of retirement is 0.25.

   (iii) The number of pensions that the city will provide on behalf of each new hire is independent of the number of pensions it will provide on behalf of any other new hire.

   Determine the probability that the city will provide at most 90 pensions to the 100 new hires and their husbands.

   (A) 0.60 (B) 0.67 (C) 0.75 (D) 0.93 (E) 0.99

   Answer: E: 0.99

   Solution: The key here is to understand that each of the 100 recruits contributes either 0, 1, or 2 pensions. The case of 0 pensions occurs when the employee leaves the force before retirement and has probability $1 - 0.4 = 0.6$. The case of 1 pension occurs if the recruit stays in the force until retirement, but remains unmarried; this case has probability $0.4 \cdot 0.25 = 0.1$. The remaining case of 2 pensions occurs when the recruit stays in the force until retirement and is married at retirement; this case has probability $1 - 0.6 - 0.1 = 0.3$.

Thus, if $X_1, \dots, X_{100}$ denote the numbers of pensions contributed by the 100 recruits, then each X_i is a random variable with the above distribution, and mean and variance given by

$$\mu = E(X_i) = 0 \cdot 0.6 + 1 \cdot 0.1 + 2 \cdot 0.3 = 0.7,$$
$$\sigma^2 = 0^2 \cdot 0.6 + 1^2 \cdot 0.1 + 2^2 \cdot 0.3 - \mu^2 = 1.3 - 0.7^2 = 0.81.$$

The total number, say X, of pensions that the city has to provide then is the sum of these X_i's. By the CLT, X is approximately normal with mean $100 \cdot 0.70 = 70$ and standard deviation $\sqrt{100} \cdot \sqrt{0.81} = 9$. Thus,

$$P(X \le 90) \approx P\left(\frac{X-70}{9} \le \frac{90-70}{9}\right) = \Phi(2.22) = 0.986.$$

10. [5-102]

For Company A there is a 60% chance that no claim is made during the coming year. If one or more claims are made, the total claim amount is normally distributed with mean 10,000 and standard deviation 2,000. For Company B there is a 70% chance that no claim is made during the coming year. If one or more claims are made, the total claim amount is normally distributed with mean 9,000 and standard deviation 2,000. Assume that the total claim amounts of the two companies are independent. What is the probability that, in the coming year, Company B's total claim amount will exceed Company A's total claim amount?

(A) 0.180 (B) 0.185 (C) 0.217 (D) 0.223 (E) 0.240

Answer: D: 0.223

Solution: What makes this problem tricky is the fact that the claim amounts, say X and Y, of the two companies only take effect if there are claims, and that for each company there is a positive probability that there are no claims.

We first carry out the calculation under the assumption that both companies do have claims. The probability that the claim amounts of B exceed those of A then is $P(Y > X)$, or, equivalently, $P(X - Y < 0)$. Since X and Y are independent normal r.v.'s with respective means 10,000 and 9,000 and variances $2,000^2$, $X - Y$ has normal distribution with mean $10,000 - 9,000 = 1000$, variance $2 \cdot 2,000^2 = 8,000,000$, and standard deviation $\sqrt{8,000,000} = 2,828$. Converting to standard units, we get

$$\begin{aligned} P(X - Y < 0) &= P\left(\frac{X - Y - 1000}{2828} < \frac{0 - 1000}{2828}\right) \\ &= P(Z < -0.3535) = P(Z > 0.3535) = 1 - P(Z \le 0.3535) \\ &= 1 - \Phi(0.3535) = 0.362. \end{aligned}$$

To account for the fact that one or both of the companies can have no claims, consider the four cases: (1) A and B both have no claims; (2) A has a claim, B has no claim; (3) A has no claim, B has a claim; and (4) A and B both have claims. These cases are mutually exclusive and exhaust all possibilities. Clearly, the claim amount paid to B exceeds that paid to B if and only if we are in case (2) (which occurs with probability $0.6 \cdot 0.3 = 0.18$) or we are in case (4) (which has probability $0.3 \cdot 0.4 = 0.12$) and the r.v.'s X and Y above satisfy $P(Y > X)$ (which occurs with probability 0.362 by the above calculation). Hence the probability in question is $0.18 + 0.12 \cdot 0.362 = 0.223$.