Math 370/408, Spring 2008
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Actuarial Exam Practice Problem Set 4

About this problem set: These are problems from Course 1/P actuarial exams that I have collected over the years, grouped by topic, and ordered by difficulty. All of the exams have appeared on the SOA website `http://www.soa.org/` at some point in the past, though most of them are no longer there. The exams are copy-righted by the SOA.

Note on the topics covered: This problem set covers problems on joint (multivariate) distributions (Chapter 4 of Hogg/Tanis).

Note on the ordering of the problems: The problems are loosely grouped by topic, and very roughly ordered by difficulty (though that ordering is very subjective). The last group of problems (those with triple digit numbers) are harder, either because they are conceptually more difficult, or simply because they require lengthy computations and may take two or three times as long to solve than an average problem.

Answers and solutions: I will post an answer key and hints/solutions on the course webpage, `www.math.uiuc.edu/~hildebr/370`.

1. [4-1]
   Let X and Y be discrete random variables with joint probability function

$$p(x,y) = \begin{cases} \frac{2x+y}{12} & \text{for } (x,y) = (0,1),(0,2),(1,2),(1,3), \\ 0 & \text{otherwise.} \end{cases}$$

   Determine the marginal probability function of X.

   (A) $p(x) = \begin{cases} 1/6 & \text{for } x = 0, \\ 5/6 & \text{for } x = 1, \\ 0 & \text{otherwise.} \end{cases}$

   (B) $p(x) = \begin{cases} 1/4 & \text{for } x = 0, \\ 3/4 & \text{for } x = 1, \\ 0 & \text{otherwise.} \end{cases}$

   (C) $p(x) = \begin{cases} 1/3 & \text{for } x = 0, \\ 2/3 & \text{for } x = 1, \\ 0 & \text{otherwise.} \end{cases}$

   (D) $p(x) = \begin{cases} 2/9 & \text{for } x = 1, \\ 3/9 & \text{for } x = 2, \\ 4/9 & \text{for } x = 3, \\ 0 & \text{otherwise.} \end{cases}$

   (E) $p(x) = \begin{cases} y/12 & \text{for } x = 0, \\ (2+y)/12 & \text{for } x = 1, \\ 0 & \text{otherwise.} \end{cases}$

2. [4-2]
   A car dealership sells 0, 1, or 2 luxury cars on any day. When selling a car, the dealer also tries to persuade the customer to buy an extended warranty for the car. Let X denote the number of luxury cars sold in a given day, and let Y denote the number of extended warranties sold, and suppose that

$$P(X=x, Y=y) = \begin{cases} 1/6 & \text{for } (x,y) = (0,0), \\ 1/12 & \text{for } (x,y) = (1,0), \\ 1/6 & \text{for } (x,y) = (1,1), \\ 1/12 & \text{for } (x,y) = (2,0), \\ 1/3 & \text{for } (x,y) = (2,1), \\ 1/6 & \text{for } (x,y) = (2,2). \end{cases}$$

   What is the variance of X?

   (A) 0.47 (B) 0.58 (C) 0.83 (D) 1.42 (E) 2.58

3. [4-3]
   An actuary determines that the annual numbers of tornadoes in counties P and Q are jointly distributed as follows:

		Annual number of tornadoes in county Q			
		0	1	2	3
Annual number	0	0.12	0.06	0.05	0.02
of tornadoes	1	0.13	0.15	0.12	0.03
in county P	2	0.05	0.15	0.10	0.02

Calculate the conditional variance of the annual number of tornadoes in county Q, given that there are no tornadoes in county P.

(A) 0.51 (B) 0.84 (C) 0.88 (D) 0.99 (E) 1.76

4. [4-4]
A diagnostic test for the presence of a disease has two possible outcomes: 1 for disease present and 0 for disease not present. Let X denote the disease state of a patient, and let Y denote the outcome of the diagnostic test. The joint probability function of X and Y is given by:

$$P(X = x, Y = y) = \begin{cases} 0.800 & \text{for } (x, y) = (0, 0), \\ 0.050 & \text{for } (x, y) = (1, 0), \\ 0.025 & \text{for } (x, y) = (0, 1), \\ 0.125 & \text{for } (x, y) = (1, 1). \end{cases}$$

Calculate $\text{Var}(Y|X = 1)$.

(A) 0.13 (B) 0.15 (C) 0.20 (D) 0.51 (E) 0.71

5. [4-5]
Let X and Y be random losses with joint density function

$$f(x, y) = e^{-x-y} \quad \text{for } x > 0 \text{ and } y > 0.$$

An insurance policy is written to reimburse $X + Y$. Calculate the probability that the reimbursement is less than 1.

(A) e^{-2} (B) e^{-1} (C) $1 - e^{-1}$ (D) $1 - 2e^{-1}$ (E) $1 - 2e^{-2}$

6. [4-6]
The waiting time for the first claim from a good driver and the waiting time for the first claim from a bad driver are independent and follow exponential distributions with means 6 years and 3 years, respectively. What is the probability that the first claim from a good driver will be filed within 3 years and the first claim from a bad driver will be filed within 2 years?

(A) $\frac{1}{18}\left(1 - e^{-2/3} - e^{-1/2} + e^{-7/6}\right)$
(B) $\frac{1}{18}e^{-7/6}$
(C) $1 - e^{-2/3} - e^{-1/2} + e^{-7/6}$

(D) $1 - e^{-2/3} - e^{-1/2} + e^{-1/3}$
(E) $1 - \frac{1}{3}e^{-2/3} - \frac{1}{6}e^{-1/2} + \frac{1}{18}e^{-7/6}$

7. [4-7]
A company has two electric generators. The time until failure for each generator follows an exponential distribution with mean 10. The company will begin using the second generator immediately after the first one fails. What is the variance of the total time that the generators produce electricity?

(A) 10 (B) 20 (C) 50 (D) 100 (E) 200

8. [4-8]
A joint density function is given by

$$f(x,y) = \begin{cases} kx & \text{for } 0 < x < 1,\ 0 < y < 1, \\ 0 & \text{otherwise.} \end{cases}$$

where k is a constant. What is $\mathrm{Cov}(X,Y)$?

(A) $-1/6$ (B) 0 (C) 1/9 (D) 1/6 (E) 2/3

9. [4-9]
A device contains two components. The device fails if either component fails. The joint density function of the lifetimes of the components, measured in hours, is $f(s,t)$, where $0 < s < 1$ and $0 < t < 1$. What is the probability that the device fails during the first half hour of operation?

(A) $\int_0^{0.5}\int_0^{0.5} f(s,t)dsdt$
(B) $\int_0^{1}\int_0^{0.5} f(s,t)dsdt$
(C) $\int_{0.5}^{1}\int_{0.5}^{1} f(s,t)dsdt$
(D) $\int_0^{0.5}\int_0^{1} f(s,t)dsdt + \int_0^{1}\int_0^{0.5} f(s,t)dsdt$
(E) $\int_0^{0.5}\int_{0.5}^{1} f(s,t)dsdt + \int_0^{1}\int_0^{0.5} f(s,t)dsdt$

10. [4-10]
Let X and Y be continuous random variables with joint density function

$$f(x,y) = \begin{cases} 15y & \text{for } x^2 \le y \le x, \\ 0 & \text{otherwise.} \end{cases}$$

Let g be the marginal density function of Y. Which of the following represents g?

(A) $g(y) = \begin{cases} 15y & \text{for } 0 < y < 1, \\ 0 & \text{otherwise.} \end{cases}$

(B) $g(y) = \begin{cases} 15y^2/2 & \text{for } x^2 < y < x, \\ 0 & \text{otherwise.} \end{cases}$

(C) $g(y) = \begin{cases} 15y^2/2 & \text{for } 0 < y < 1, \\ 0 & \text{otherwise.} \end{cases}$

(D) $g(y) = \begin{cases} 15y^{3/2}(1 - y^{1/2}) & \text{for } x^2 < y < x, \\ 0 & \text{otherwise.} \end{cases}$

(E) $g(y) = \begin{cases} 15y^{3/2}(1 - y^{1/2}) & \text{for } 0 < y < 1, \\ 0 & \text{otherwise.} \end{cases}$

11. [4-11]

Let X represent the age of an insured automobile involved in an accident. Let Y represent the length of time the owner has insured the automobile at the time of the accident. X and Y have joint probability density function

$$f(x, y) = \begin{cases} \frac{1}{64}(10 - xy^2) & \text{for } 2 \le x \le 10 \text{ and } 0 \le y \le 1, \\ 0 & \text{otherwise.} \end{cases}$$

Calculate the expected age of an insured automobile involved in an accident.

(A) 4.9 (B) 5.2 (C) 5.8 (D) 6.0 (E) 6.4

12. [4-12]

Let X and Y be continuous random variables with joint density function

$$f(x, y) = \begin{cases} 24xy & \text{for } 0 < x < 1 \text{ and } 0 < y < 1 - x, \\ 0 & \text{otherwise.} \end{cases}$$

Calculate $P(Y < X | X = 1/3)$.

(A) 1/27 (B) 2/27 (C) 1/4 (D) 1/3 (E) 4/9

13. [4-13]

Let X and Y be random losses with joint density function

$$f(x, y) = 2x \quad \text{for } 0 < x < 1 \text{ and } 0 < y < 1.$$

An insurance policy is written to cover the loss $X + Y$. The policy has a deductible of 1. Calculate the expected payment under the policy.

(A) 1/4 (B) 1/3 (C) 1/2 (D) 7/12 (E) 5/6

14. [4-14]

Let T_1 and T_2 represent the lifetimes in hours of two linked components in an electronic device. The joint density function for T_1 and T_2 is uniform over the region defined by $0 \le t_1 \le t_2 \le L$, where L is a positive constant. Determine the expected value of the sum of the squares of T_1 and T_2.

(A) $L^2/3$ (B) $L^2/2$ (C) $2L^2/3$ (D) $3L^2/4$ (E) L^2

15. [4-15]

A company offers a basic life insurance policy to its employees, as well as a supplemental life insurance policy. To purchase the supplemental policy, an employee must first purchase the basic policy. Let X denote the proportion of employees who purchase the basic policy, and Y the proportion of employees who purchase the supplemental policy. Let X and Y have the joint density function $f(x, y) = 2(x + y)$ on the region where the density is positive. Given that 10% of the employees buy the basic policy, what is the probability that fewer than 5% buy the supplemental policy?

(A) 0.010 (B) 0.013 (C) 0.108 (D) 0.417 (E) 0.500

16. [4-16]

The future lifetimes (in months) of two components of a machine have the following joint density function:

$$f(x,y) = \begin{cases} \frac{6}{125{,}000}(50 - x - y) & \text{for } 0 < x < 50 - y < 50, \\ 0 & \text{otherwise.} \end{cases}$$

What is the probability that both components are still functioning 20 months from now?

(A) $\dfrac{6}{125,000} \displaystyle\int_0^{20} \int_0^{20} (50 - x - y)dydx$

(B) $\dfrac{6}{125,000} \displaystyle\int_{20}^{30} \int_{20}^{50-x} (50 - x - y)dydx$

(C) $\dfrac{6}{125,000} \displaystyle\int_{20}^{30} \int_{20}^{50-x-y} (50 - x - y)dydx$

(D) $\dfrac{6}{125,000} \displaystyle\int_{20}^{50} \int_{20}^{50-x} (50 - x - y)dydx$

(E) $\dfrac{6}{125,000} \displaystyle\int_{20}^{50} \int_{20}^{50-x-y} (50 - x - y)dydx$

17. [4-17]

A device contains two circuits. The second circuit is a backup for the first, so the second is used only when the first has failed. The device fails when and only when the second circuit fails. Let X and Y be the times at which the first and second circuits fail, respectively. X and Y have joint probability density function

$$f(x,y) = \begin{cases} 6e^{-x}e^{-2y} & \text{for } 0 < x < y < \infty, \\ 0 & \text{otherwise.} \end{cases}$$

What is the expected time at which the device fails?

(A) 0.33 (B) 0.50 (C) 0.67 (D) 0.83 (E) 1.50

18. [4-51]

Once a fire is reported to a fire insurance company, the company makes an initial estimate, X, of the amount it will pay to the claimant for the fire loss. When the claim is finally settled, the company pays an amount, Y, to the claimant. The company has determined that X and Y have the joint density function

$$f(x,y) = \frac{2}{x^2(x-1)} y^{-(2x-1)/(x-1)}, \quad x > 1, y > 1.$$

Given that the initial claim estimated by the company is 2, determine the probability that the final settlement amount is between 1 and 3.

(A) 1/9 (B) 2/9 (C) 1/3 (D) 2/3 (E) 8/9

19. [4-53]

A company is reviewing tornado damage claims under a farm insurance policy. Let X be the portion of a claim representing damage to the house and let Y be the portion of the same claim representing damage to the rest of the property. The joint density function of X and Y is

$$f(x,y) = \begin{cases} 6(1-(x+y)) & \text{for } x>0, y>0, x+y<1, \\ 0 & \text{otherwise.} \end{cases}$$

Determine the probability that the portion of a claim representing damage to the house is less than 0.2.

(A) 0.360 (B) 0.480 (C) 0.488 (D) 0.512 (E) 0.520

20. [4-54]

An auto insurance policy will pay for damage to both the policyholder's car and the other driver's car in the event that the policyholder is responsible for an accident. The size of the payment for damage to the policyholder's car, X, has a marginal density function of 1 for $0 < x < 1$. Given $X = x$, the size of the payment for damage to the other driver's car, Y, has conditional density of 1 for $x < y < x+1$. If the policyholder is responsible for an accident, what is the probability that the payment for damage to the other driver's car will be greater than 0.500?

(A) 3/8 (B) 1/2 (C) 3/4 (D) 7/8 (E) 15/16

21. [4-55]

An insurance policy is written to cover a loss X, where X has density function

$$f(x) = \begin{cases} \frac{3}{8}x^2 & \text{for } 0 \le x \le 2, \\ 0 & \text{otherwise.} \end{cases}$$

The time (in hours) to process a claim of size x, where $0 \le x \le 2$, is uniformly distributed on the interval from x to $2x$. Calculate the probability that a randomly chosen claim on this policy is processed in three hours or more.

(A) 0.17 (B) 0.25 (C) 0.32 (D) 0.58 (E) 0.83

22. [4-56]

Two insurers provide bids on an insurance policy to a large company. The bids must be between 2000 and 2200. The company decides to accept the lower bid if the two bids differ by 20 or more. Otherwise, the company will consider the two bids further. Assume that the two bids are independent and are both uniformly distributed on the interval from 2000 to 2200. Determine the probability that the company considers the two bids further.

(A) 0.10 (B) 0.19 (C) 0.20 (D) 0.41 (E) 0.60

23. [4-102]

Let X and Y be continuous random variables with joint density function

$$f(x,y) = \begin{cases} \frac{8}{3}xy & \text{for } 0 \le x \le 1,\ x \le y \le 2x, \\ 0 & \text{otherwise.} \end{cases}$$

Calculate the covariance of X and Y.

(A) 0.04 (B) 0.25 (C) 0.67 (D) 0.80 (E) 1.24

24. [4-103]

A device runs until either of two components fails, at which point the device stops running. The joint density function of the lifetimes of the two components, both measured in hours, is

$$f(x,y) = \begin{cases} \frac{1}{27}(x+y) & \text{for } 0 < x < 3 \text{ and } 0 < y < 3, \\ 0 & \text{otherwise.} \end{cases}$$

Calculate the probability that the device fails during its first hour of operation.

(A) 0.04 (B) 0.41 (C) 0.44 (D) 0.59 (E) 0.96

25. [4-104]

A family buys two policies from the same insurance company. Losses under the two policies are independent and have continuous uniform distributions on the interval from 0 to 10. One policy has a deductible of 1 and the other has a deductible of 2. The family experiences exactly one loss under each policy. Calculate the probability that the total benefit paid to the family does not exceed 5.

(A) 0.13 (B) 0.25 (C) 0.30 (D) 0.32 (E) 0.42

26. [4-106]

An insurance company sells two types of auto insurance policies: Basic and Deluxe. The time until the next Basic Policy claim is an exponential random variable with mean two days. The time until the next Deluxe Policy claim is an independent exponential random variable with mean three days. What is the probability that the next claim will be a Deluxe Policy claim?

(A) 0.172 (B) 0.223 (C) 0.400 (D) 0.487 (E) 0.500

27. [4-107]

An insurance company insures a large number of drivers. Let X be the random variable representing the company's losses under collision insurance, and let Y represent the company's losses under liability insurance. X and Y have joint density function

$$f(x,y) = \begin{cases} \frac{1}{4}(2x+2-y) & \text{for } 0 < x < 1 \text{ and } 0 < y < 2, \\ 0 & \text{otherwise.} \end{cases}$$

What is the probability that the total loss is at least 1?

(A) 0.33 (B) 0.38 (C) 0.41 (D) 0.71 (E) 0.75

28. [4-108]

The stock prices of two companies at the end of any given year are modeled with random variables X and Y that follow a distribution with joint density function

$$f(x,y) = \begin{cases} 2x & \text{for } 0 < x < 1,\ x < y < x+1, \\ 0 & \text{otherwise.} \end{cases}$$

What is the conditional variance of Y given that $X = x$?

(A) $1/12$ (B) $7/6$ (C) $x + 1/2$ (D) $x^2 - 1/6$ (E) $x^2 + x + 1/3$

29. [4-109]

Claim amounts for wind damage to insured homes are independent random variables with common density function

$$f(x) = \begin{cases} 3x^{-4} & \text{for } x > 1, \\ 0 & \text{otherwise.} \end{cases}$$

where x is the amount of a claim in thousands. Suppose 3 such claims will be made. What is the expected value of the largest of the three claims?

(A) 2025 (B) 2700 (C) 3232 (D) 3375 (E) 4500

30. [4-110]

A company insures homes in three cities, J, K, L. The losses occurring in these cities are independent. The moment-generating functions for the loss distributions of the cities are

$$M_J(t) = (1-2t)^{-3}, \quad M_K(t) = (1-2t)^{-2.5}, \quad M_L(t) = (1-2t)^{-4.5}.$$

Let X represent the combined losses from the three cities. Calculate $E(X^3)$.

(A) $1,320$ (B) $2,082$ (C) $5,760$ (D) $8,000$ (E) $10,560$

31. [4-111]

An insurance contract reimburses a family's automobile accident losses up to a maximum of two accidents per year. The joint probability distribution for the number of accidents of a three person family (X,Y,Z) is $p(x,y,z) = k(x+2y+z)$, where $x = 0,1$, $y = 0,1,2$, $z = 0,1,2$, and x,y,z are the numbers of accidents incurred by X, Y, Z, respectively. Determine the expected number of unreimbursed accident losses given that X is not involved in any accidents.

(A) $5/21$ (B) $1/3$ (C) $5/9$ (D) $46/63$ (E) $7/9$

32. [4-112]

Suppose the remaining lifetimes of a husband and wife are independent and uniformly distributed on the interval $[0,40]$. An insurance company offers two products to married couples: one which pays when the husband dies, and one which pays when both the husband and wife have died. Calculate the covariance of the two payment times.

(A) 0.0 (B) 44.4 (C) 66.7 (D) 200.0 (E) 466.7

33. [4-115]

Let X and Y denote the values of two stocks at the end of a five-year period. X is uniformly distributed on the interval $(0, 12)$. Given $X = x$, Y is uniformly distributed on the interval $(0, x)$. Determine $\text{Cov}(X, Y)$ according to this model.

(A) 0 (B) 4 (C) 6 (D) 12 (E) 24

34. [4-113]

A company offers earthquake insurance. Annual premiums are modeled by an exponential random variable with mean 2. Annual claims are modeled by an exponential random variable with mean 1. Premiums and claims are independent. Let X denote the ratio of claims to premiums. What is the density function of X?

(A) $\frac{1}{2x+1}$ (B) $\frac{2}{(2x+1)^2}$ (C) e^{-x} (D) $2e^{-2x}$ (E) xe^{-x}