Math 370/408, Spring 2008
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Actuarial Exam Practice Problem Set 3 Solutions

About this problem set: These are problems from Course 1/P actuarial exams that I have collected over the years, grouped by topic, and ordered by difficulty. All of the exams have appeared on the SOA website `http://www.soa.org/` at some point in the past, though most of them are no longer there. The exams are copy-righted by the SOA.

Note on the topics covered: This problem set covers problems on continuous random variables (Chapter 3 of Hogg/Tanis).

Note on the ordering of the problems: The problems are loosely grouped by topic, and very roughly ordered by difficulty (though that ordering is very subjective). The last group of problems (those with triple digit numbers) are harder, either because they are conceptually more difficult, or simply because they require lengthy computations and may take two or three times as long to solve than an average problem.

Answers and solutions: I will post an answer key and hints/solutions on the course webpage, `www.math.uiuc.edu/~hildebr/370`.

1. [3-1]

An insurance company issued insurance policies to 32 independent risks. For each policy, the probability of a claim is 1/6. The benefit given that there is a claim has probability density function

$$f(y) = \begin{cases} 2(1-y), & 0 < y < 1, \\ 0, & \text{otherwise.} \end{cases}$$

Calculate the expected value of total benefits paid.

(A) $\frac{16}{9}$ (B) $\frac{8}{3}$ (C) $\frac{32}{9}$ (D) $\frac{16}{3}$ (E) $\frac{32}{3}$

Answer: A: 16/9

Hint/Solution: The expected value of a single benefit is $E(Y) = \int_0^1 yf(y)dy = \int_0^1 (2y - 2y^2)dy = 1/3$. The expected number of claims is $32 \cdot (1/6) = 16/3$. Multiplying these two numbers gives the expected value of all benefits paid, $(1/3)(16/3) = 16/9$.

2. [3-2]

Let X be a continuous random variable with density function

$$f(x) = \begin{cases} \frac{|x|}{10} & \text{for } -2 \le x \le 4, \\ 0 & \text{otherwise.} \end{cases}$$

Calculate the expected value of X.

(A) $\frac{1}{5}$ (B) $\frac{3}{5}$ (C) 1 (D) $\frac{28}{15}$ (E) $\frac{12}{5}$

Answer: D: 28/15

Hint/Solution: This is an easy integration exercise. The only tricky part here is the correct handling of the absolute value $|x|$ appearing in the definition of $f(x)$: One has to split the integral into two parts, corresponding to ranges $-2 \le x \le 0$ and $0 < x \le 4$, replacing $|x|$ by $-x$ in the first range, and by x in the second range, before evaluating the integral.

3. [3-3]

A large company has determined that the function

$$f(x) = 3x^2, \quad 0 \le x \le 1,$$

serves as the payroll density function. That is, the distribution of payroll, $F(x)$, which is the proportion of total payroll earned by the lowest paid fraction x of employees, $0 \le x \le 1$, relates to $f(x)$ in the same way that probability distributions and probability densities relate. Gini's index, G, defined as

$$G = 2\int_0^1 |x - F(x)|dx$$

is a measure of how evenly payroll is distributed among all employees. Calculate G for this large company.

(A) 0.2 (B) 0.4 (C) 0.5 (D) 0.8 (E) 1.0

Answer: C: 0.5

Hint/Solution: This reduces to an easy integration problem. First compute $F(x)$ by integrating $f(x)$, then compute the integral in the definition of G.

4. [3-4]

The loss amount, X, for a medical insurance policy has cumulative distribution function

$$F(x) = \begin{cases} 0, & x < 0, \\ \frac{1}{9}\left(2x^2 - \frac{x^3}{3}\right), & 0 \le x \le 3, \\ 1, & x > 3. \end{cases}$$

Calculate the mode of the distribution.

(A) $\frac{2}{3}$ (B) 1 (C) $\frac{3}{2}$ (D) 2 (E) 3

Answer: D: 2

Hint/Solution: The mode of a distribution is the point where $f(x)$ is maximal. With the given distribution we have $f(x) = F'(x) = (4/9)x - (1/9)x^2$ for $0 < x < 3$, and $f(x) = 0$ outside the interval $[0, 3]$. Differentiating, we get $f'(x) = (4/9) - (2/9)x$, and setting this equal to 0, we see that $x = 2$ is the only critical point of f. Hence the mode must occur at $x = 2$.

5. [3-5]

An insurance company's monthly claims are modeled by a continuous, positive random variable X, whose probability density function is proportional to $(1+x)^{-4}$, where $0 < x < \infty$. Determine the company's expected monthly claims.

(A) $\frac{1}{6}$ (B) $\frac{1}{3}$ (C) $\frac{1}{2}$ (D) 1 (E) 3

Answer: C: 1/2

Hint/Solution: We have $f(x) = c(1+x)^{-4}$ for $0 < x < \infty$, where c is a constant. First determine the proportionality constant c by setting $\int_0^\infty c(1+x)^{-4}$ equal to 1 and solving for c. This gives $c = 3$. Then compute the integral $E(X) = \int_0^\infty xf(x)dx = \int_0^\infty 3x(1+x)^{-4}dx$ by substituting $u = 1 + x$, $du = dx$.

6. [3-6]

An insurance policy reimburses dental expense, X, up to a maximum benefit of 250. The probability density function for X is

$$f(x) = \begin{cases} ce^{-0.004x} & \text{for } x \ge 0, \\ 0 & \text{otherwise}, \end{cases}$$

where c is a constant. Calculate the median benefit for this policy.

(A) 161 (B) 165 (C) 173 (D) 182 (E) 250

Answer: C: 173

Hint/Solution: The given distribution has the form of an exponential distribution, so the constant c must be 0.004, and the c.d.f. is equal to $F(x) = 1 - e^{-0.004x}$. To obtain the median, set $F(x) = 0.5$ and solve for x: $1 - e^{-0.004x} = 0.5$, so $-0.004x = \ln 0.5$, and $x = -\ln(0.5)/0.004 = 173$.

7. [3-7]

The time to failure of a component in an electronic device has an exponential distribution with a median of four hours. Calculate the probability that the component will work without failing for at least five hours.

(A) 0.07 (B) 0.29 (C) 0.38 (D) 0.42 (E) 0.57

Answer: D: 0.42

Hint/Solution: The general form of the c.d.f. of an exponential distribution is $F(x) = 1 - e^{-x/\theta}$ for $x > 0$. Since the median is 4 hours, we have $F(4) = 1/2$. This allows us to determine the parameter θ: $1 - e^{-4/\theta} = 1/2$, so $\theta = 4/\ln 2$. The probability that the component will work for at least 5 hours is given by $1 - F(5) = e^{-5/\theta} = e^{-(5/4)\ln 2} = 0.42$.

8. [3-8]

The number of days that elapse between the beginning of a calendar year and the moment a high-risk driver is involved in an accident is exponentially distributed. An insurance company expects that 30% of high-risk drivers will be involved in an accident during the first 50 days of a calendar year. What portion of high-risk drivers are expected to be involved in an accident during the first 80 days of a calendar year?

(A) 0.15 (B) 0.34 (C) 0.43 (D) 0.57 (E) 0.66

Answer: C: 0.43

Hint/Solution: If X denotes the number of days elapsed until an accident occurs, then we are given that X is exponentially distributed, and that $P(X \leq 50) = 0.3$, and we have to compute $P(X \leq 80)$. By the general of the c.d.f. of an exponential distribution, we have $F(x) = P(X \leq x) = 1 - e^{-x/\theta}$ for $x > 0$. By the given information, $F(50) = P(X \leq 50) = 0.3$, which allows us to determine θ: We have $1 - e^{-50/\theta} = 0.3$, so $-50/\theta = \ln 0.7$, or $\theta = -50/\ln 0.7$. Substituting this back into $F(x)$ and setting $x = 80$, we get $F(80) = P(X \leq 80) = 1 - e^{-80/\theta} = 1 - e^{(80/50)\ln 0.7} = 0.43$

9. [3-9]

The monthly profit of Company I can be modeled by a continuous random variable with density function f. Company II has a monthly profit that is twice that of Company I. Determine the probability density function of the monthly profit of Company II.

(A) $\frac{1}{2}f\left(\frac{x}{2}\right)$ (B) $f\left(\frac{x}{2}\right)$ (C) $2f\left(\frac{x}{2}\right)$ (D) $2f(x)$ (E) $2f(2x)$

Answer: A

Hint/Solution: This is an easy exercise in changing variables in density functions if you apply to correct procedure for that. Let X denote the profit for Company I and Y that for Company II. Then $Y = 2X$. Let $f(x)$ and $F(x)$ denote the p.d.f. and c.d.f. of X, and let $g(x)$ and $G(x)$ denote the corresponding functions for Y. Then we have $G(x) = P(Y \leq x) = P(2X \leq x) = P(X \leq x/2) = F(x/2)$. Differentiating both sides with respect to x we get $g(x) = G'(x) = (1/2)F'(x/2) = (1/2)f(x/2)$ since $F'(x) = f(x)$. Thus, the density of Y is $(1/2)f(x/2)$.

10. [3-10]

An insurance company insures a large number of homes. The insured value, X, of a randomly selected home is assumed to follow a distribution with density function

$$f(x) = \begin{cases} 3x^{-4} & \text{for } x > 1, \\ 0 & \text{otherwise.} \end{cases}$$

Given that a randomly selected home is insured for at least 1.5, what is the probability that it is insured for less than 2?

(A) 0.578 (B) 0.684 (C) 0.704 (D) 0.829 (E) 0.875

Answer: A: 0.578

Hint/Solution: We need to compute

$$P(X \leq 2|X \geq 1.5) = \frac{P(1.5 \leq X \leq 2)}{P(X > 1.5)}.$$

Now,

$$P(X > 1.5) = \int_{1.5}^{\infty} 3x^{-4} = 1.5^{-3} = 1.5^{-3} = 0.296,$$

and

$$P(1.5 \leq X \leq 2) = P(X > 1.5) - P(X > 2) = 1.5^{-3} - 2^{-3} = 0.171,$$

so $P(X \leq 2|X \geq 1.5) = 0.171/0.296 = 0.578$.

11. [3-11]

An insurance policy reimburses a loss up to a benefit limit of 10. The policyholder's loss, Y, follows a distribution with density function

$$f(y) = \begin{cases} 2y^{-3} & \text{for } y > 1, \\ 0 & \text{otherwise.} \end{cases}$$

What is the expected value of the benefit paid under the insurance policy?

(A) 1.0 (B) 1.3 (C) 1.8 (D) 1.9 (E) 2.0

Answer: D: 1.9

Hint/Solution: The benefit, X, is given by $X = Y$ if $Y \leq 10$, and by $X = 10$ if $Y > 10$. Thus,

$$E(X) = \int_1^{10} y \cdot 2y^{-3} dy + \int_{10}^{\infty} 10 \cdot 2y^{-3} dy = 2\left(1 - \frac{1}{10}\right) + 10 \cdot 10^{-2} = 1.9$$

12. [3-12]

The lifetime of a machine part has a continuous distribution on the interval $(0, 40)$ with probability density function f, where $f(x)$ is proportional to $(10 + x)^{-2}$. Calculate the probability that the lifetime of the machine part is less than 6.

(A) 0.04 (B) 0.15 (C) 0.47 (D) 0.53 (E) 0.94

Answer: C: 0.47

Hint/Solution: We are given that $f(x) = c(10 + x)^{-2}$ for $0 \leq x \leq 40$, with some constant c. To determine c, compute the integral of $f(x)$ over the full interval $(0, 40)$ and set the integral equal to 1: $1 = \int_0^{40} c(10 + x)^{-x} dx = c((1/10) - (1/50)) = c0.08$, so $c = 1/0.08 = 12.5$. The probability to compute is then equal to $\int_0^6 c(10 + x)^{-2} dx = c((1/10) - (1/16)) = c(3/80) = 0.47$.

13. [3-13]

The time, T, that a manufacturing system is out of operation has cumulative distribution function

$$F(t) = \begin{cases} 1 - \left(\frac{2}{t}\right)^2 & \text{for } t > 2, \\ 0 & \text{otherwise.} \end{cases}$$

The resulting cost to the company is $Y = T^2$. Determine the density function of Y, for $y > 4$.

(A) $4y^{-2}$ (B) $8y^{-3/2}$ (C) $8y^{-3}$ (D) $16y^{-1}$ (E) $1024y^{-5}$

Answer: A

Hint/Solution: This is a routine application of the change of variables techniques; as always in these problems, you need to take a detour via the corresponding c.d.f.'s to get the p.d.f. of the new variable.

14. [3-14]

Under a group insurance policy, an insurer agrees to pay 100% of the medical bills incurred during the year by employees of a small company, up to a maximum total of one million dollars. The total amount of bills incurred, X, has probability density function

$$f(x) = \begin{cases} \frac{x(4-x)}{9} & \text{for } 0 < x < 3, \\ 0 & \text{otherwise,} \end{cases}$$

where x is measured in millions. Calculate the total amount, in millions of dollars, the insurer would expect to pay under this policy.

(A) 0.120 (B) 0.301 (C) 0.935 (D) 2.338 (E) 3.495

Answer: C: 0.935

Hint/Solution: Let Y denote the payout (in millions). Then $Y = X$ if $X \le 1$, and $Y = 1$ otherwise. Thus,

$$\begin{aligned} E(Y) &= \int_0^1 x f(x)dx + \int_1^\infty 1 \cdot f(x)dx \\ &= \int_0^1 \frac{1}{9}x^2(4-x)dx + \int_1^3 \frac{1}{9}x(4-x)dx \\ &= \frac{1}{9}\left(4\frac{1}{3} - \frac{1}{4}\right) + \frac{1}{9}\left(\frac{3^2-1^2}{2} - \frac{3^3-1^3}{3}\right) \\ &= \frac{13}{108} + \frac{22}{27} = 0.935. \end{aligned}$$

15. [3-15]

The lifetime of a printer costing 200 is exponentially distributed with mean 2 years. The manufacturer agrees to pay a full refund to a buyer if the printer fails during the first year following its purchase, and a one-half refund if it fails during the second year. If the manufacturer sells 100 printers, how much should it expect to pay in refunds?

(A) $6,321$ (B) $7,358$ (C) $7,869$ (D) $10,256$ (E) $12,642$

Answer: D: 10,256

Hint/Solution: Let X denote the lifetime of the printer. Then the refund for each printer, Y, is given by $Y = 200$ if $X \le 1$, $Y = 100$ if $1 \le X \le 2$, and $Y = 0$ otherwise. Thus, $E(Y) = 200P(X \le 1) + 100P(1 < X \le 2)$. The probabilities $P(\dots)$ here are easily computed using that fact X has exponential distribution with mean 2, and thus c.d.f. $F(x) = P(X \le x) = 1 - e^{-x/2}$ for $x \ge 0$. Thus, $P(X \le 1) = 1 - e^{-1/2}$, $P(1 < X \le 2) = P(X \le 2) - P(X \le 1) = (1 - e^{-2/2}) - (1 - e^{-1/2}) = e^{-1/2} - e^{-1}$, and hence $E(Y) = 200 \cdot (1 - e^{-1/2}) + 100 \cdot (e^{-1/2} - e^{-1}) = 102.5$. Multiplying by the number of printers, 100, gives the total refund, $10,250$.

16. [3-16]

An insurance policy is written to cover a loss, X, where X has a uniform distribution on $[0, 1000]$. At what level must a deductible be set in order for the expected payment to be 25% of what it would be with no deductible?

(A) 250 (B) 375 (C) 500 (D) 625 (E) 750

Answer: C: 500

Hint/Solution: First note, that without the deductible the expected payment would be 500 (since it is then equal to the loss, and the loss X is uniformly distributed on $[0, 1000]$). Next, let D denote the (unknown) deductible and Y the payment under deductible D. Then $Y = X - D$ if $D \le X \le 1000$, and $Y = 0$ if $X \le D$. Thus,

$$E(Y) = \int_D^{1000} (x-D)f(x)dx = \int_D^{1000} (x-D)\frac{1}{1000}dx = \frac{1}{1000}\cdot\frac{(1000-D)^2}{2} = \frac{(1000-D)^2}{2000}$$

Now set this expression equal to 25% of 500, i.e., 125, and solve for D: $(1000-D)^2/2000 = 125$, or $(1000 - D)^2 = 250000$, so $D = 500$.

17. [3-17]

An insurance company sells an auto insurance policy that covers losses incurred by a policyholder, subject to a deductible of 100. Losses incurred follow an exponential distribution with mean 300. What is the 95th percentile of actual losses that exceed the deductible?

(A) 600 (B) 700 (C) 800 (D) 900 (E) 1000

Answer: E: 1000

Hint/Solution: The main difficulty here is the correct interpretation of the "95th percentile of actual losses that exceed the deductible". The proper interpretation involves a conditional probability: we seek the value x such that the conditional probability that the loss is at most x, given that it exceeds the deductible, is 0.95, i.e., that $P(X \le x|X \ge 100) = 0.95$, where X denotes the loss. By the complement formula for conditional probabilities, this is equivalent to $P(X \ge x|X \ge 100) = 0.05$. Since X is exponentially distributed with mean 300, we have $P(X \ge x) = e^{-x/300}$, so for $x > 100$,

$$P(X \ge x|X \ge 100) = \frac{P(X \ge x)}{P(X \ge 100} = \frac{e^{-x/300}}{e^{-100/300}} = e^{-(x-100)/300}.$$

Setting this equal to 0.05 and solving for x, we get $(x - 100)/300 = -\ln(0.05)$, so $x = -300\ln(0.05) + 100 = 1000$.

18. [3-18]

A piece of equipment is being insured against early failure. The time from purchase until failure of the equipment is exponentially distributed with mean 10 years. The insurance will pay an amount x if the equipment fails during the first year, and it will pay $0.5x$ if failure occurs during the second or third year. If failure occurs after the first three years, no payment will be made. At what level must x be set if the expected payment made under this insurance is to be 1000?

(A) 3858 (B) 4449 (C) 5382 (D) 5644 (E) 7235

Answer: D: 5644

Hint/Solution: Let T be the time until failure and X the payout. Then $X = x$ if $T \leq 1$, $X = (1/2)x$ if $1 \leq T \leq 3$, and $X = 0$ otherwise. (Note that x here is not the variable in a p.d.f. or c.d.f., but the unknown given in the problem that we need to determine.) Thus, $E(X) = xP(T \leq 1) + (1/2)xP(1 \leq T \leq 3)$. Now $P(T \leq t) = 1 - e^{-t/10}$ by the given exponential distribution for T. Thus, $P(T \leq 1) = 1 - e^{-1/10} = 0.096$ and $P(1 \leq T \leq 3) = e^{-1/10} - e^{-3/10} = 0.163$. Substituting these values into $E(X)$ gives $E(X) = 0.096x + 0.5 \cdot 0.163x = 0.177x$. Setting this equal to 1000 and solving for x we get the answer, $x = 1000/0.1777 = 5644$.

19. [3-19]

A manufacturer's annual losses follow a distribution with density function

$$f(x) = \begin{cases} \frac{2.5(0.6)^{2.5}}{x^{3.5}} & \text{for } x > 0.6, \\ 0 & \text{otherwise.} \end{cases}$$

To cover its losses, the manufacturer purchases an insurance policy with an annual deductible of 2. What is the mean of the manufacturer's annual losses not paid by the insurance policy?

(A) 0.84 (B) 0.88 (C) 0.93 (D) 0.95 (E) 1.00

Answer: C: 0.93

Hint/Solution: Let X denote the annual loss, and Y the part of the loss not covered by insurance. Then $Y = X$ if $X \leq 2$, and $Y = 2$ if $X \geq 2$. To compute $E(Y)$, split the integral into intervals $[0.6, 2]$ and $[2, \infty]$.

20. [3-20]

A device that continuously measures and records seismic activity is placed in a remote region. The time, T, to failure of this device is exponentially distributed with mean 3 years. Since the device will not be monitored during its first two years of service, the time to discovery of its failure is $X = \max(T, 2)$. Determine $E(X)$.

(A) $2 + \frac{1}{3}e^{-6}$
(B) $2 - 2e^{-2/3} + 5e^{-4/3}$
(C) 3
(D) $2 + 3e^{-2/3}$
(E) 5

Answer: D

Hint/Solution: Since T is exponentially distributed with mean 3, the density of T is $f(t) = (1/3)e^{-t/3}$ for $t > 0$. Since $X = \max(T, 2)$, we have $X = 2$ if $0 \le T \le 2$ and $X = T$ if $2 < T < \infty$.

Thus,

$$\begin{aligned} E(X) &= \int_0^2 2\frac{1}{3}e^{-t/3}dt + \int_2^\infty t \cdot \frac{1}{3}e^{-t/3}dt \\ &= 2(1 - e^{-2/3}) - te^{-t/3}\Big|_2^\infty + \int_2^\infty e^{-t/3}dt \\ &= 2(1 - e^{-2/3}) + 2e^{-t/3} + 3e^{-2/3} = 2 + 3e^{-2/3} \end{aligned}$$

21. [3-21]

An insurance policy pays for a random loss X subject to a deductible of C, where $0 < C < 1$. The loss amount is modeled as a continuous random variable with density function

$$f(x) = \begin{cases} 2x & \text{for } 0 < x < 1, \\ 0 & \text{otherwise.} \end{cases}$$

Given a random loss X, the probability that the insurance payment is less than 0.5 is equal to 0.64. Calculate C.

(A) 0.1 (B) 0.3 (C) 0.4 (D) 0.6 (E) 0.8

Answer: B: 0.3

Hint/Solution: The insurance payment, Y, is given by $Y = X - C$ if $X \ge C$, and $Y = 0$ if $X < C$. Thus, $Y \le 0.5$ is equivalent to $X \le 0.5 + C$, and we have

$$P(Y \le 0.5) = P(X \le C + 0.5) = \int_0^{C+0.5} 2x dx = (C + 0.5)^2.$$

Setting this equal to 0.64 and solving for C we get $(C + 0.5)^2 = 0.64$, so $C = \sqrt{0.64} - 0.5 = 0.3$. (Note that this argument would not be correct if C were greater than 0.5 since then the upper limit in the integral falls outside the range of the density. However, it is easy to see that this case cannot occur: Since the given density function restricts X to the interval $0 \le X \le 1$, if $C > 1/2$, then $P(X \le 0.5 + C) = 1$, and hence also $P(Y \le 0.5) = 1$, contrary to the given condition $P(Y \le 0.5) = 0.64$.

22. [3-22]

An actuary is reviewing a study she performed on the size of the claims made ten years ago under homeowners insurance policies. In her study, she concluded that the size of claims followed an exponential distribution and that the probability that a claim would be less than \$1,000 was 0.250. The actuary feels that the conclusions she reached in her study are still valid today with one exception: every claim made today would be twice the size of a similar claim made ten years ago as a result of inflation. Calculate the probability that the size of a claim made today is less than \$1,000.

Answer: C: 0.134

Hint/Solution: First determine the mean θ from the equation $F(1000) = 0.25$, then compute $F(500)$ (note that 500 is the claim size today corresponding to a claim size of 1000 ten years ago).

23. [3-23]

A group insurance policy covers the medical claims of the employees of a small company. The value, V, of the claims made in one year is described by $V = 100,000Y$, where Y is a random variable with density function

$$f(y) = \begin{cases} k(1-y)^4 & \text{for } 0 < y < 1, \\ 0 & \text{otherwise,} \end{cases}$$

where k is a constant. What is the conditional probability that V exceeds $40,000$, given that V exceeds $10,000$?

(A) 0.08 (B) 0.13 (C) 0.17 (D) 0.20 (E) 0.51

Answer: B: 0.13

Hint/Solution: First determine k by setting the integral over $f(y)$ equal to 1: $1 = \int_0^1 f(y) = k\int_0^1 (1-y)^4 dy = k(1/5)$, so $k = 5$. The conditional probability to compute is

$$P(Y \geq 40,000|Y \geq 10,000) = P(V \geq 0.4|V \geq 0.1) = \frac{P(V \geq 0.4)}{P(V \geq 0.1)}.$$

Since V has density $f(y) = 5(1-y)^4$ for $0 < y < 1$, $P(V \geq 0.4) = \int_{0.4}^1 5(1-y)^4 = -((1-1)^5 - (1-0.4)^5) = 0.6^5$ and similarly $P(V \geq 0.1) = 0.9^5$, so the conditional probability sought is $0.6^5/0.9^5 = 0.13$.

24. [3-24]

The value, ν, of an appliance is given by $\nu(t) = e^{7-0.2t}$, where t denotes the number of years since purchase. If appliance fails within seven years of purchase, a warranty pays the owner the value of the appliance at the time of failure. After seven years, the warranty pays nothing. The time until failure of the appliance has an exponential distribution with mean 10. Calculate the expected payment from the warranty.

(A) 98.70 (B) 109.66 (C) 270.43 (D) 320.78 (E) 352.16

Answer: D: 320.78

Hint/Solution: Let X denote the payout under the warranty, and T the time until failure. Then $X = e^{7-0.2T}$ if $T \leq 7$, and $X = 0$ otherwise. Since T has exponential density with mean 10, its density function is $(1/10)e^{-t/10}$ for $t > 0$. Thus,

$$\begin{aligned} E(X) &= \int_0^7 e^{7-0.2t}\frac{1}{10}e^{-t/10}dt \\ &= 0.1 \cdot e^7 \int_0^7 e^{-0.3t}dt = 0.1 \cdot e^7 \frac{e^{-0.3\cdot 7} - 1}{-0.3} = 320.78 \end{aligned}$$

25. [3-51]

The loss due to a fire in a commercial building is modeled by a random variable X with density function

$$f(x) = \begin{cases} 0.005(20-x) & \text{for } 0 < x < 20, \\ 0 & \text{otherwise.} \end{cases}$$

Given that a fire loss exceeds 8, what is the probability that it exceeds 16?

(A) 1/25 (B) 1/9 (C) 1/8 (D) 1/3 (E) 3/7

Answer: B: 1/9

Hint/Solution: This is and easy integration exercise. We need to compute $P(X \geq 16|X \geq 8)$, which is the same as $P(X \geq 16)/P(X \geq 8)$. Integrating the given density function, we get

$$P(X \geq x) = \int_x^{20} 0.005(20 - t)dt = 0.0025(20 - x)^2, \quad 0 < x < 20.$$

Taking the ratio of these expressions with $x = 16$ and $x = 8$ gives the answer: $(20 - 16)^2/(20 - 8)^2 = 1/9$.

26. [3-52]

An insurer's annual weather-related loss, X, is a random variable with density function

$$f(x) = \begin{cases} \dfrac{2.5(200)^{2.5}}{x^{3.5}} & \text{for } x > 200, \\ 0 & \text{otherwise.} \end{cases}$$

Calculate the difference between the 30th and 70th percentiles of X.

(A) 35 (B) 93 (C) 124 (D) 231 (E) 298

Answer: B: 93

Hint/Solution: We first compute the c.d.f. $F(x)$ of X by integrating the given density:

$$F(x) = \int_{200}^{x} 2.5(200)^{2.5} t^{-3.5} dt = 1 - \left(\frac{200}{x}\right)^{2.5}, \quad x > 200.$$

Setting $F(x)$ equal to 0.3 and 0.7 and solving for x we get the 30th and 70th percentiles: $x = 230.7$ and $x = 323.7$. The answer is the difference between these two numbers, 93.

27. [3-53]

The warranty on a machine specifies that it will be replaced at failure or age 4, whichever occurs first. The machine's age at failure, X, has density function

$$f(x) = \begin{cases} 1/5 & \text{for } 0 < x < 5, \\ 0 & \text{otherwise.} \end{cases}$$

Let Y be the age of the machine at the time of replacement. Determine the variance of Y.

(A) 1.3 (B) 1.4 (C) 1.7 (D) 2.1 (E) 7.5

Answer: C: 1.7

Hint/Solution: We have

$$Y = \begin{cases} X & \text{if } 0 < X \leq 5, \\ 4 & \text{if } 4 < X \leq 5. \end{cases}$$

Thus,

$$\begin{aligned}
E(Y) &= \int_0^4 x\frac{1}{5}dt + \int_4^5 4\frac{1}{5}dt = 2.4, \\
E(Y^2) &= \int_0^4 x^2\frac{1}{5}dt + \int_4^5 4^2\frac{1}{5}dt = 7.47 \\
\mathrm{Var}(Y) &= E(Y^2) - E(Y)^2 = 1.707.
\end{aligned}$$

28. [3-54]

The owner of an automobile insures it against damage by purchasing an insurance policy with a deductible of 250. In the event that the automobile is damaged, repair costs can be modeled by a uniform random variable on the interval $(0, 1500)$. Determine the standard deviation of the insurance payment in the event that the automobile is damaged.

(A) 361 (B) 403 (C) 433 (D) 464 (E) 521

Answer: B: 403

Hint/Solution: Let X denote the repair costs, and let Y denote the insurance payment. We need to compute $\sigma = \sqrt{\mathrm{Var}(Y)}$.

We are given that X is uniformly distributed on the interval $(0, 1500)$, so the density of X is $f(x) = 1/1500$ for $0 < x < 1500$. Taking into account the 250 deductible, we see that Y is given by

$$Y = \begin{cases} X - 250 & \text{if } 250 \le X \le 1500, \\ 0 & \text{otherwise.} \end{cases}$$

Hence,

$$\begin{aligned}
E(Y) &= \int_{250}^{1500} (x - 250)\frac{1}{1500}dx = \frac{1250^2}{2 \cdot 1500}, \\
E(Y^2) &= \int_{250}^{1500} (x - 250)^2\frac{1}{1500}dx = \frac{1250^3}{3 \cdot 1500}, \\
\sigma &= \sqrt{E(Y^2) - E(Y)^2} = 403.3.
\end{aligned}$$

29. [3-55]

An actuary models the lifetime of a device using the random variable $Y = 10X^{0.8}$, where X is an exponential random variable with mean 1 year. Determine the probability density function $f(y)$, for $y > 0$, of the random variable Y.

(A) $10y^{0.8}e^{-8y^{-0.2}}$

(B) $8y^{-0.2}e^{-10y^{0.8}}$

(C) $8y^{-0.2}e^{-0.1y^{1.25}}$

(D) $(0.1y)^{1.25}e^{-0.125(0.1y)^{0.25}}$

(E) $0.125(0.1y)^{0.25}e^{-(0.1y)^{1.25}}$

Answer: E

Hint/Solution: This is a change of variables exercise. Let $g(x)$ and $G(x)$ denote the density and c.d.f. of X, and let $f(y)$ and $F(y)$ denote the density and c.d.f. of Y. Then, for $y > 0$,

$$\begin{aligned} F(y) &= P(Y \le y) = P(10X^{0.8} \le y) \\ &= P(X \le (0.1y)^{1.25}) = G((0.1y)^{1.25}). \end{aligned}$$

Differentiating with respect to y we get

$$\begin{aligned} f(y) = F'(y) &= G'((0.1y)^{1.25})1.25(0.1y)^{0.25}0.1 \\ &= g((0.1y)^{1.25})0.125(0.1y)^{0.25} = e^{-(0.1y)^{1.25}}0.125(0.1y)^{0.25}. \end{aligned}$$

since $g(x) = e^{-x}$ by the exponential distribution of X with mean 1.

30. [3-56]

A tour operator has a bus that can accommodate 20 tourists. The operator knows that tourists may not show up, so he sells 21 tickets. The probability that an individual tourist will not show up is 0.02, independent of all other tourists. Each ticket costs 50, and is non-refundable if a tourist fails to show up. If a tourist shows up and a seat is not available, the tour operator has to pay 100 (ticket cost + 50 penalty) to the tourist. What is the expected revenue of the tour operator?

(A) 935 (B) 950 (C) 967 (D) 976 (E) 985

Answer: E:985

Hint/Solution: This is simpler than it might seem since there are only two cases to consider:

(i) All 21 ticket holders show up. This occurs with probability $(1 - 0.02)^{21} = 0.654$. In this case, the operator earns $21 \cdot 50 = 1050$ from ticket sales, but incurs a cost of 100 for the one ticket holder that cannot be accommodated, so the revenue is $1050 - 100 = 950$.

(ii) Not all 21 ticket holders show up. This occurs with the complementary probability, $1 - 0.654 = 0.346$. In this case, the operator earns 1050 from ticket sales, but incurs no penalty, so the revenue is 1050.

Hence the expected revenue is $950 \cdot 0.654 + 1050 \cdot 0.346 = 985$.

31. [3-57]

An investment account earns an annual interest rate R that follows a uniform distribution on the interval $(0.04, 0.08)$. The value of a $10,000$ initial investment in this account after one year is given by $V = 10,000e^R$. Determine the cumulative distribution function, $F(v)$, of V for values v that satisfy $0 < F(v) < 1$.

(A) $\dfrac{10,000e^{v/10,000} - 10,408}{425}$

(B) $25e^{v/10,000} - 0.04$

(C) $\dfrac{v - 10,408}{10,833 - 10,408}$

(D) $\dfrac{25}{v}$

(E) $25\left(\ln\left(\frac{v}{10,000}\right) - 0.04\right)$

Answer: E

Hint/Solution: This is a change of variables exercise. Let $g(r)$ and $G(r)$ denote the density and c.d.f. of R, and let $f(v)$ and $F(v)$ denote the density and c.d.f. of Y. Then, for v in the appropriate range,

$$\begin{aligned} F(v) &= P(V \le v) = P(10000e^R \le v) \\ &= P\left(R \le \ln\left(\frac{v}{10000}\right)\right) = G\left(\ln\left(\frac{v}{10000}\right)\right). \end{aligned}$$

Since R has uniform distribution on the interval $(0.04, 0.08)$, $G(r)$ is given by (for r in this interval)

$$G(r) = \frac{r - 0.04}{0.08 - 0.04} = 25(r - 0.04).$$

Substituting this above gives the correct answer, (E):

$$F(v) = 25\left(\ln\left(\frac{v}{10,000}\right) - 0.04\right).$$

32. [3-61]

You are given the following information about N, the annual number of claims for a randomly selected insured:

$$P(N=0) = \frac{1}{2}, \quad P(N=1) = \frac{1}{3}, \quad P(N>1) = \frac{1}{6}.$$

Let S denote the total annual claim amount for an insured. When $N = 1$, S is exponentially distributed with mean 5. When $N > 1$, S is exponentially distributed with mean 8. Determine $P(4 < S < 8)$.

(A) 0.04 (B) 0.08 (C) 0.12 (D) 0.24 (E) 0.25

Answer: C

Hint/Solution: First note that, if S were a pure exponential variable with a *fixed* mean θ and c.d.f. $F(x)$, then

$$P(4 < S < 8) = F(8) - F(4) = (1 - e^{-8/\theta}) - (1 - e^{-4/\theta}) = e^{-4/\theta} - e^{-8/\theta}.$$

For the particular means $\theta = 5$ and $\theta = 8$, these probabilities become 0.0823 and 0.0397. The probability sought in the problem is a weighted average of these two probabilities, the weigths being the probabilities with which these two cases occur: 1/3 for mean 5 and 1/6 for mean 8. The result is $0.0823(1/3) + 0.0397(1/6) = 0.122$.

33. [3-102]

Let T denote the time in minutes for a customer service representative to respond to 10 telephone inquiries. T is uniformly distributed on the interval with endpoints 8 minutes and 12 minutes. Let R denote the average rate, in customers per minute, at which the representative responds to inquiries. Which of the following is the density function of the random variable R on the interval $(10/12 \le r \le 10/8)$?

(A) $\frac{12}{5}$

(B) $3 - \frac{5}{2r}$

(C) $3r - \frac{5\ln r}{2}$

(D) $\frac{10}{r^2}$

(E) $\frac{5}{2r^2}$

Answer: E

Hint/Solution: The key observation here is that R and T are related by $R = 10/T$. Since we know the distribution of T (uniform distribution on $[8, 12]$), the problem then becomes a change of variables problem for density functions. As usual, this requires a detour through the corresponding c.d.f.'s. Let $f(t)$ and $F(t)$ denote the p.d.f. and c.d.f. of T, and let $g(r)$ and $G(r)$ denote the corresponding functions for R. By the given uniform distribution of T on $[8, 12]$, we have $f(t) = 1/4$ for $8 \le t \le 12$. Hence, for $10/12 \le r \le 10/8$,

$$G(r) = P(R \le r) = P\left(\frac{10}{T} \le r\right) = P\left(T \ge \frac{10}{r}\right) = 1 - F\left(\frac{10}{r}\right).$$

Differentiating, we get

$$g(r) = G'(r) = -F'\left(\frac{10}{r}\right)(-10r^{-2}) = f\left(\frac{10}{r}\right)10r^{-2} = \frac{1}{4}10r^{-2} = \frac{5}{2r^2}.$$

34. [3-103]

An auto insurance company insures an automobile worth $15,000$ for one year under a policy with a $1,000$ deductible. During the policy year there is a 0.04 chance of partial damage to the car and a 0.02 chance of a total loss of the car. If there is partial damage to the car, the amount X of damage (in thousands) follows a distribution with density function

$$f(x) = \begin{cases} 0.5003e^{-x/2} & \text{for } 0 < x < 15, \\ 0 & \text{otherwise.} \end{cases}$$

What is the expected claim payment?

(A) 320 (B) 328 (C) 352 (D) 380 (E) 540

Answer: B: 328

Hint/Solution: This is a very lengthy computation.

Let Y be the claim payment in thousands, and let D and T stand for partial damage and total loss, respectively. We need to compute $E(Y)$. To this end we need to consider separately the case when there there is a partial loss and the case when there is a total loss, and compute the conditional expectation of Y under these case cases, $E(Y|D)$ and $E(Y|T)$. The overall expectation then is given by $E(Y) = E(Y|D)P(D) + E(Y|T)P(T)$.

We are given that $P(D) = 0.04$ and $P(T) = 0.02$. Moreover, in the case of a total loss, the payment is (in thousands) $15 - 1 = 14$, so $E(Y|T) = 14$. Thus, it remains to compute $E(Y|D)$, the expected loss in case in case of partial damage. From the problem, we have in this case

$$Y = \begin{cases} X - 1 & \text{if } 1 \le X < 15, \\ 0 & \text{otherwise.} \end{cases}$$

Thus,

$$\begin{aligned}
E(Y|D) &= \int_1^{15} (x-1)f(x)dx = \int_1^{15} (x-1)0.5003e^{-x/2}dx \\
&= \Big[(-2)(x-1)0.5003e^{-x/2}\Big]_1^{15} + 2\int_1^{15} 0.5003e^{-x/2}dx \\
&= (-2)(14)0.5003e^{-15/2} + 2\cdot 2\cdot 0.5003\left(e^{-1/2} - e^{-15/2}\right) \\
&= 1.2049
\end{aligned}$$

Hence,

$$E(Y) = 1.2049 \cdot 0.04 + 14 \cdot 0.02 = 0.328.$$