

Math 370/408, Spring 2008
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Actuarial Exam Practice Problem Set 2
Solutions

About this problem set: These are problems from Course 1/P actuarial exams that I have collected over the years, grouped by topic, and ordered by difficulty. All of the exams have appeared on the SOA website <http://www.soa.org/> at some point in the past, though most of them are no longer there. The exams are copy-righted by the SOA.

Note on the topics covered: This problem set covers discrete random variables (Chapter 2 of Hogg/Tanis).

Note on the ordering of the problems: The problems are loosely grouped by topic, and very roughly ordered by difficulty (though that ordering is very subjective). The last group of problems (those with triple digit numbers) are harder, either because they are conceptually more difficult, or simply because they require lengthy computations and may take two or three times as long to solve than an average problem.

Answers and solutions: I will post an answer key and hints/solutions on the course webpage, www.math.uiuc.edu/~hildebr/370.

1. [2-1]

An insurance policy pays an individual 100 per day for up to 3 days of hospitalization and 25 per day for each day of hospitalization thereafter. The number of days of hospitalization, X , is a discrete random variable with probability function

$$P(X = k) = \begin{cases} \frac{6-k}{15} & \text{for } k = 1, 2, 3, 4, 5, \\ 0 & \text{otherwise.} \end{cases}$$

Calculate the expected payment for hospitalization under this policy.

- (A) 85 (B) 163 (C) 168 (D) 213 (E) 255

Answer: D: 213

Solution: A fairly easy exercise in discrete densities; just make a table of the *total* payment for each value of k , along with the associated probability $P(X = k)$, and then compute the expected value of this payment.

2. [2-2]

An actuary determines that the claim size for a certain class of accidents is a random variable, X , with moment generating function

$$M_X(t) = \frac{1}{(1 - 2500t)^4}.$$

Determine the standard deviation of the claim size for this class of accidents.

- (A) 1,340 (B) 5,000 (C) 8,660 (D) 10,000 (E) 11,180

Answer: B: 5,000

Solution: Use the formulas $M'(0) = E(X)$, $M''(0) = E(X^2)$.

3. [2-3]

An actuary has discovered that policyholders are three times as likely to file two claims as to file four claims. If the number of claims filed has a Poisson distribution, what is the variance of the number of claims filed?

- (A) $1/\sqrt{3}$ (B) 1 (C) $\sqrt{2}$ (D) 2 (E) 4

Answer: D: 2

Solution: Let X denote the number of claims. We are given that X has Poisson distribution and that $P(X = 2) = 3P(X = 4)$. Substitute the formula for a Poisson p.m.f., $P(X = x) = e^{-\lambda}\lambda^x/x!$ into this equation to determine λ . Then use the fact that $\sigma^2 = \lambda$ for a Poisson distribution with parameter λ .

4. [2-4]

In modeling the number of claims filed by an individual under an automobile policy during a three-year period, an actuary makes the simplifying assumption that for all integers $n \geq 0$, $p_{n+1} = (1/5)p_n$, where p_n represents the probability that the policyholder files n claims during the period. Under this assumption, what is the probability that a policyholder files more than one claim during the period?

- (A) 0.04 (B) 0.16 (C) 0.20 (D) 0.80 (E) 0.96

Answer: A: 0.04

Solution: The given relation for p_n implies that $p_n = (1/5)p_{n-1} = (1/5)^2 p_{n-2} = \dots = (1/5)^n p_0$ for $n = 0, 1, 2, \dots$. Since these probabilities must add up to 1, we have $1 = \sum_{n=0}^{\infty} (1/5)^n p_0 = (1 - 1/5)^{-1} p_0 = (5/4)p_0$ (using the formula for the sum of a geometric series), so $p_0 = 4/5$. The probability to compute is $1 - p_0 - p_1 = 1 - 4/5 - 4/25 = 0.04$.

5. [2-5]

As part of the underwriting process for insurance, each prospective policyholder is tested for high blood pressure. Let X represent the number of tests completed when the first person with high blood pressure is found. The expected value of X is 12.5. Calculate the probability that the sixth person tested is the first one with high blood pressure.

- (A) 0.000 (B) 0.053 (C) 0.080 (D) 0.316 (E) 0.394

Answer: B: 0.053

Solution: By the formulas for a geometric distribution, $\mu = 1/p$, so $p = 1/\mu = 1/12.5$, and $P(X = 6) = (1 - p)^5 p = (11.5/12.5)^5 (1/12.5) = 0.053$.

6. [2-6]

Let X be a random variable with moment generating function

$$M(t) = \left(\frac{2 + e^t}{3} \right)^9, \quad -\infty < t < +\infty.$$

Calculate the variance of X .

- (A) 2 (B) 3 (C) 8 (D) 9 (E) 11

Answer: A: 2

Solution: Use the formulas $E(X) = M'(0)$ and $E(X^2) = M''(0)$. Be careful when computing the derivatives of $M(t)$, using the chain rule!

7. [2-7]

A small commuter plane has 30 seats. The probability that any particular passenger will not show up for a flight is 0.10, independent of other passengers. The airline sells 32 tickets for the flight. Calculate the probability that more passengers show up for the flight than there are seats available.

- (A) 0.0042 (B) 0.0343 (C) 0.0382 (D) 0.1221 (E) 0.1564

Answer: E: 0.1564

Solution: Consider the 32 tickets as 32 Bernoulli (success/failure) trials with success meaning that the passenger holding the ticket shows up (so that $p = 0.9$). The probability to compute is then that of getting 31 or 32 successes in 32 such trials, which is given by the binomial distribution.

8. [2-8]

The distribution of loss due to fire damage to a warehouse is:

Amount of loss	Probability
0	0.900
500	0.060
1,000	0.030
10,000	0.008
50,000	0.001
100,000	0.001

Given that a loss is greater than zero, calculate the expected amount of the loss.

- (A) 290 (B) 322 (C) 1,704 (D) 2,900 (E) 32,222

Answer: D: 2,900

Solution: This would be a straightforward calculation except for the phrase “given that the loss is greater than 0”. The correct interpretation of this condition is to restrict consideration to those cases where the loss is greater than 0 and renormalize the probabilities for these cases so that they again add up to 1. Since the probability for a loss of 0 is 0.9, the remaining probabilities must add up to 0.1, so dividing each by 0.1 renormalizes them to have sum 1. Using these renormalized probabilities in the formula for an expectation gives the desired expectation under the above condition.

9. [2-9]

Let X_1, X_2, X_3 be a random sample from a discrete distribution with probability function

$$p(x) = \begin{cases} \frac{1}{3} & \text{for } x = 0, \\ \frac{2}{3} & \text{for } x = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Determine the moment generating function, $M(t)$, of $Y = X_1X_2X_3$.

- (A) $\frac{19}{27} + \frac{8}{27}e^t$
 (B) $1 + 2e^t$
 (C) $\left(\frac{1}{3} + \frac{2}{3}e^t\right)^3$
 (D) $\frac{1}{27} + \frac{8}{27}e^{3t}$
 (E) $\frac{1}{3} + \frac{2}{3}e^{3t}$

Answer: A

Solution: This is easier than it looks at first glance, since $Y = X_1X_2X_3$ takes on only values 0 and 1, and $Y = 1$ occurs if and only if all of X_1, X_2, X_3 are equal to 1. The latter occurs with probability $(2/3)^3 = 8/27$, $P(Y = 1) = 8/27$ and $P(Y = 0) = 1 - 8/27 = 19/27$, and therefore $M(t) = e^{0t}P(Y = 0) + e^{1t}P(Y = 1) = 1 \cdot (19/27) + e^t \cdot (8/27)$.

10. [2-51]

The number of injury claims per month is modeled by a random variable N with

$$P(N = n) = \frac{1}{(n+1)(n+2)}, \quad \text{where } n \geq 0.$$

Determine the probability of at least one claim during a particular month, given that there have been at most four claims during that month.

- (A) $1/3$ (B) $2/5$ (C) $1/2$ (D) $3/5$ (E) $5/6$

Answer: B: $2/5$

Solution: This is a very easy problem: We need to compute $P(N \geq 1 | N \leq 4)$, which is the same as $P(1 \leq N \leq 4) / P(N \leq 4)$. Now,

$$P(1 \leq N \leq 4) = \sum_{n=1}^4 P(N = n) = \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \frac{1}{30} = \frac{1}{3},$$

$$P(N \leq 4) = \sum_{n=0}^4 P(N = n) = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \frac{1}{30} = \frac{5}{6}.$$

Dividing the first probability by the second gives the answer: $(1/3)/(5/6) = 2/5$.

11. [2-52]

An insurance company determines that N , the number of claims received in a week, is a random variable with $P(N = n) = 2^{-n-1}$, where $n \geq 0$. The company also determines that the number of claims received in a given week is independent of the number of claims received in any other week. Determine the probability that exactly seven claims will be received during a given two-week period.

- (A) $\frac{1}{256}$ (B) $\frac{1}{128}$ (C) $\frac{7}{512}$ (D) $\frac{1}{64}$ (E) $\frac{1}{32}$

Answer: D: $1/64$

Solution: Let N_1 and N_2 , denote the number of claims during the first, respectively second, week. The total number of claims received during the two-week period then is $N_1 + N_2$, and so we need to compute $P(N_1 + N_2 = 7)$. By breaking this probability down into cases according to the values of (N_1, N_2) , and using the independence of N_1 and N_2 and the given distribution, we get

$$P(N_1 + N_2 = 7) = \sum_{n=0}^7 P(N_1 = n, N_2 = 7 - n) = \sum_{n=0}^7 2^{-n-1} 2^{-(7-n)-1}$$

$$= \sum_{n=0}^7 2^{-9} = 8 \cdot 2^{-9} = \frac{1}{64}.$$

12. [2-53]

A company prices its hurricane insurance using the following assumptions:

- (i) In any calendar year, there can be at most one hurricane.
- (ii) In any calendar year, the probability of a hurricane is 0.05.
- (iii) The number of hurricanes in any calendar year is independent of the number of hurricanes in any other calendar year.

Using the company's assumptions, calculate the probability that there are fewer than 3 hurricanes in a 20-year period.

- (A) 0.06 (B) 0.19 (C) 0.38 (D) 0.62 (E) 0.92

Answer: E: 0.92

Solution: By the given assumptions the occurrence of hurricanes can be modeled as success/failure trials, with success meaning that a hurricane occurs in a given year and occurring with probability $p = 0.05$. Thus, the probability that there are fewer than 3 hurricanes in a 20-year period is equal to the probability of having less than 3 successes in 20 success/failure trials with $p = 0.05$. By the binomial distribution, this probability is

$$\begin{aligned} \sum_{x=0}^2 \binom{20}{x} 0.05^x 0.95^{20-x} \\ = 0.95^{20} + 20 \cdot 0.05 \cdot 0.95^{19} + \frac{20 \cdot 19}{2} \cdot 0.05^2 \cdot 0.95^{18} = 0.92. \end{aligned}$$

13. [2-54]

An insurance policy on an electrical device pays a benefit of 4000 if the device fails during the first year. The amount of the benefit decreases by 1000 each successive year until it reaches 0. If the device has not failed by the beginning of any given year, the probability of failure during that year is 0.4. What is the expected benefit under this policy?

- (A) 2234 (B) 2400 (C) 2500 (D) 2667 (E) 2694

Answer: E: 2694

Solution: Let X denote the year in which the device fails, and Y the benefit. Then

$$Y = \begin{cases} 4000 & \text{if } X = 1, \\ 3000 & \text{if } X = 2, \\ 2000 & \text{if } X = 3, \\ 1000 & \text{if } X = 4, \\ 0 & \text{if } X \geq 5. \end{cases}$$

We need to compute $E(Y)$. This is easy once we know the probabilities $P(X = n)$ for $n = 1, 2, 3, 4$. Now $P(X = n)$ is the probability that the device is working during years $1, 2, \dots, n - 1$ and fails during year n . If we assume that these probabilities behave like success/failure probabilities in a sequence of independent success/failure trials, then we would have $P(X = n) = (1 - p)^{n-1}p$, with p , the failure probability in a given year, equal to 0.4, so

$$P(X = n) = 0.6^{n-1}0.4, \quad n = 1, 2, \dots \quad (*)$$

With this assumption we get

$$E(Y) = 4,000 \cdot 0.4 + 3,000 \cdot 0.6 \cdot 0.4 + 2,000 \cdot 0.6^2 \cdot 0.4 + 1,000 \cdot 0.6^3 \cdot 0.4 = 2694.$$

which is the correct answer.

The above argument, while leading to the correct answer, requires additional justification since the above assumption, while natural, is not directly stated in the problem. Instead the assumption made in the problem is contained in the phrase: *If the device has not failed by the beginning of any given year, the probability of failure during that year is 0.4*. In terms of

the random variable X (the year of failure), this translates into $P(X = n|X \geq n) = 0.4$, or, equivalently (**) $P(X = n)/P(X \geq n) = 0.4$. From this we can derive the probabilities (*): Setting $n = 1, 2, 3, \dots$ in (**), we get $P(X = 1) = 0.4P(X \geq 1) = 0.4 \cdot 1 = 0.4$, $P(X = 2) = 0.4P(X \geq 2) = 0.4(1 - P(X = 1)) = 0.4(1 - 0.4) = 0.4 \cdot 0.6$, $n = 3$, argument gives $P(X = 3) = 0.4(1 - 0.4 - 0.4 \cdot 0.6) = 0.4 \cdot 0.6^2$, etc.

14. [2-55]

An insurance company sells a one-year automobile policy with a deductible of 2. The probability that the insured will incur a loss is 0.05. If there is a loss, the probability of a loss of amount N is K/N , for $N = 1, \dots, 5$ and K a constant. These are the only possible loss amounts and no more than one loss can occur. Determine the net premium for this policy.

- (A) 0.031 (B) 0.066 (C) 0.072 (D) 0.110 (E) 0.150

Answer: A: 0.031

Solution: The “net premium” is the premium (per policy) at which the company breaks even. This is equal to the expected insurance payout (per policy). Taking into account the deductible, the payout is 1 if $N = 3$; 2 if $N = 4$; 3 if $N = 5$; and 0 in all other cases. Thus, the expected payout is

$$1P(N = 3) + 2P(N = 4) + 3P(N = 5) = 1\frac{K}{3} + 2\frac{K}{4} + 3\frac{K}{5} = \frac{43}{30}K.$$

To determine the constant K , we use the fact that the (overall) probability of a loss is 0.05. Thus, the sum of the probabilities for the possible loss amounts has to equal 0.05:

$$0.05 = \sum_{N=1}^5 \frac{K}{N} = \frac{137}{60}K.$$

Hence $K = 3/137$, and the above expectation becomes $(43/30)(3/137) = 0.03138$.

15. [2-56]

A probability distribution of the claim sizes for an auto insurance policy is given in the table below:

Claim size	Probability
20	0.15
30	0.10
40	0.05
50	0.20
60	0.10
70	0.10
80	0.30

What percentage of the claims are within one standard deviation of the mean claim size?

- (A) 45% (B) 55% (C) 68% (D) 85% (E) 100%

Answer: A: 0.45

Solution: A routine, but rather lengthy computation: First find $\mu = E(X)$, $E(X^2)$, $\text{Var}(X)$, and σ . Then add up the probabilities for those x -values that satisfy $|x - \mu| \leq \sigma$.

16. [2-60]

Two life insurance policies, each with a death benefit of 10,000 and a one-time premium of 500, are sold to a couple, one for each person. The policies will expire at the end of the tenth year. The probability that only the wife will survive at least ten years is 0.025, the probability that only the husband will survive at least ten years is 0.01, and the probability that both of them will survive at least ten years is 0.96. What is the expected excess of premiums over claims, given that the husband survives at least ten years?

- (A) 350 (B) 385 (C) 397 (D) 870 (E) 897

Answer: E: 897

Solution: Let X denote the excess of premiums over claims. Then we have to compute $E(X|\text{husband survives})$, the conditional expectation of X given that the husband survives. We have two cases to consider:

- (a) Only the husband survives. This occurs with probability 0.01. In this case the claims are 10000, while the premiums collected are $2 \cdot 500 = 1000$. Thus, $X = 1000 - 10000 = -9000$.
- (b) Both husband and wife survive. This occurs with probability 0.96. In this case the claims are 0, so with the premiums being as before 1000, we have $X = 1000 - 0 = 1000$.

The probability that the husband survives is the sum of the above probabilities, i.e., $0.01 + 0.96 = 0.97$. Hence, the desired conditional expectation is

$$E(X|\text{husband survives}) = \frac{(-9000)(0.01) + (1000)(0.96)}{0.97} = 896.9$$

17. [2-101]

A company establishes a fund of 120 from which it wants to pay an amount, C , to any of its 20 employees who achieve a high performance level during the coming year. Each employee has a 2% chance of achieving a high performance level during the coming year, independent of any other employee. Determine the maximum value of C for which the probability is less than 1% that the fund will be inadequate to cover all payments for high performance.

- (A) 24 (B) 30 (C) 40 (D) 60 (E) 120

Answer: D: 60

Solution: At first glance this seems to be a Central Limit Theorem problem, but it actually is just a trial-and-error calculation with the binomial distribution. In order to determine C we need to find the minimal integer n such that the probability that there are more than n high performance employees is below 0.01. The maximal value of C then is $120/n$.

Let X denote the number of high performance employees. Then X has binomial distribution with $n = 20$ and $p = 0.02$, and we can calculate: $P(X = 0) = 0.98^{20} = 0.668$, $P(X = 1) = 20 \cdot 0.02 \cdot 0.98^{19} = 0.272$, $P(X = 2) = \binom{20}{2} \cdot 0.02^2 \cdot 0.98^{18} = 0.053$. Since the sum of all three of these probabilities, $0.668 + 0.272 + 0.053 = 0.993$, exceeds 0.99, but the sum of the first two alone, $0.668 + 0.272 = 0.94$, does not, $n = 2$ is the minimal integer such that $P(X > 2)$ is less than 0.01. Hence $C = 120/2 = 60$.

18. [2-102]

A company takes out an insurance policy to cover accidents that occur at its manufacturing plant. The probability that one or more accidents will occur during any given month is $3/5$. The number of accidents that occur in any given month is independent of the number of accidents that occur in all other months. Calculate the probability that there will be at least four months in which no accidents occur before the fourth month in which at least one accident occurs.

- (A) 0.01 (B) 0.12 (C) 0.23 (D) 0.29 (E) 0.41

Answer: D: 0.29

Solution: The given situation can be modelled as a series of independent success/failure trials, with the trials corresponding to the months, and success meaning that one or more accidents have occurred in that month, so that the success probability is $p = 3/5$. Thus, in the language of this model, we seek the probability that there will be at least 4 failures before the 4-th success occurs. Now the event “at least 4 failures occur before the 4-th success” is exactly equivalent to “4-th success occurs at trial 8 or later” (think about it!). We compute the probability of the latter event by considering its complement, “4-th success occurs at trial 7 or earlier”. The latter event splits into four disjoint events,

$$A_n = \text{“success in trial } n, \text{ exactly 3 successes in trials } 1, \dots, n-1\text{”}, \quad n = 4, 5, 6, 7.$$

Now

$$\begin{aligned} P(A_n) &= p \cdot P(3 \text{ successes in } n-1 \text{ trials}) = p \binom{n-1}{3} p^3 (1-p)^{n-4} \\ &= \binom{n-1}{3} (3/5)^4 (2/5)^{n-4}. \end{aligned}$$

Hence, the probability we seek is

$$\begin{aligned} 1 - \sum_{n=4}^7 P(A_n) &= 1 - \binom{3}{3} 0.6^4 + \binom{4}{3} 0.6^4 0.4 + \binom{5}{3} 0.6^4 0.4^2 + \binom{6}{3} 0.6^4 0.4^3 \\ &= 1 - 0.6^4 (1 + 4 \cdot 0.4 + 10 \cdot 0.4^2 + 20 \cdot 0.4^3) = 0.29. \end{aligned}$$

19. [2-103]

A company buys a policy to insure its revenue in the event of major snowstorms that shut down business. The policy pays nothing for the first such snowstorm of the year and 10,000 for each one thereafter, until the end of the year. The number of major snowstorms per year that shut down business is assumed to have Poisson distribution with mean 1.5. What is the expected amount paid to the company under this policy during a one-year period?

- (A) 2,769 (B) 5,000 (C) 7,231 (D) 8,347 (E) 10,578

Answer: C: 7,231

Solution: The main difficulty in this problem is that it requires a slightly tricky manipulation of the exponential series. Let X denote the number of snowstorms, and let Y denote the insurance payout, measured in units of 10,000. We need to calculate $E(Y)$. From the given information, X is Poisson distributed with mean 1.5 and Y is related to X by

$$Y = \begin{cases} X - 1 & \text{if } X = 1, 2, \dots, \\ 0 & \text{if } X = 0. \end{cases}$$

Hence,

$$\begin{aligned} E(Y) &= \sum_{x=1}^{\infty} (x-1)P(X=x) = \sum_{x=1}^{\infty} (x-1)e^{-1.5} \frac{1.5^x}{x!} \\ &= e^{-1.5} \left(\sum_{x=1}^{\infty} x \frac{1.5^x}{x!} - \sum_{x=1}^{\infty} \frac{1.5^x}{x!} \right) \\ &= e^{-1.5} \left(1.5 \sum_{x=1}^{\infty} \frac{1.5^{x-1}}{(x-1)!} - \left(\sum_{x=0}^{\infty} \frac{1.5^x}{x!} - 1 \right) \right) \\ &= e^{-1.5} (1.5e^{1.5} - e^{1.5} + 1) \\ &= 0.5 - e^{-1.5} = 0.7231. \end{aligned}$$

Since the units of Y are 10,000, we get 7,231 as answer.

20. [2-104]

A study is being conducted in which the health of two independent groups of ten policyholders is being monitored over a one-year period of time. Individual participants in the study drop out before the end of the study with probability 0.2 (independently of the other participants). What is the probability that at least 9 participants complete the study in one of the two groups, but not in both groups?

- (A) 0.096 (B) 0.192 (C) 0.235 (D) 0.376 (E) 0.469

Answer: E: 0.469

Solution: This is a two stage problem. For the first stage, focus on a single group, and consider the probability, say q , that in this group at least 9 participants complete the study. Then q is the probability that in 10 success/failure trials with success probability $p = 0.8$ there are at least 9 successes, so q can be computed using the binomial distribution: $q = 1 - 0.8^{10} - \binom{10}{9}0.8^9 0.2 = 0.62419$.

For the second stage, we consider two such groups, and compute the probability that exactly one of them is “successful” in the above sense. This probability is given by $2q(1-q) = 0.46939$, as can be seen either by considering separately the cases “group 1 successful, group 2 unsuccessful” and “group 1 unsuccessful, group 2 successful”, or by another application of the success/trial model, this time with 2 trials and success probability q .