Math 370, Actuarial Problemsolving
Spring 2008
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Problem Set 1 (with solutions)

About this problem set: These are problems from Course 1/P actuarial exams that I have collected over the years, grouped by topic, and ordered by difficulty. All of the exams have appeared on the SOA website `http://www.soa.org/` at some point in the past, though most of them are no longer there. The exams are copy-righted by the SOA.

Note on the topics covered: The problems are exercises in applying set-theoretic properties and rules, general rules of probability, and the definitions and properties of independence and conditional probability, Problems 1-9 through 1-15 are exercises in Bayes' Rule, which is of great practical significance and which seems to come up in virtually every actuarial exam..

Note on the ordering of the problems: The problems are loosely grouped by topic, and very roughly ordered by difficulty (though that ordering is very subjective). The last few problems of the set are harder, either because they are conceptually more difficult, or simply because they require lengthy computations and may take two or three times as long to solve than an average problem.

Answers and solutions: I will post an answer key and hints/solutions on the course webpage, `www.math.uiuc.edu/~hildebr/370`.

1. [Problem 1-1]

   A marketing survey indicates that 60% of the population owns an automobile, 30% owns a house, and 20% owns both an automobile and a house. Calculate the probability that a person chosen at random owns an automobile or a house, but not both.

   (A) 0.4 (B) 0.5 (C) 0.6 (D) 07 (E) 09

   Answer: B: 0.5

   Hint/Solution: An easy Venn diagram exercise.

2. [Problem 1-2]

   The probability that a visit to a primary care physician's (PCP) office results in neither lab work nor referral to a specialist is 35%. Of those coming to a PCP's office, 30% are referred to specialists and 40% require lab work. Determine the probability that a visit to a PCP's office results in both lab work and referral to a specialist.

   (A) 0.05 (B) 0.12 (C) 0.18 (D) 0.25 (E) 0.35

   Answer: A: 0.05

   Hint/Solution: An easy exercise in using Venn diagrams, or the formula for P(A union B).

3. [Problem 1-3]

   You are given $P(A \cup B) = 0.7$ and $P(A \cup B') = 0.9$. Determine $P(A)$.

   (A) 0.2 (B) 0.3 (C) 0.4 (D) 0.6 (E) 0.8

   Answer: D: 0.6

   Hint/Solution: Use the given data and the formula for P(A union B) to obtain two equations for P(A) and P(B), and solve.

4. [Problem 1-4]

   Workplace accidents are categorized into three groups: minor, moderate, severe. The probability that a given accident is minor is 0.5, that it is moderate is 0.4, and that it is severe is 0.1. Two accidents occur independently in one month. Calculate the probability that neither accident is severe and at most one is moderate.

   (A) 0.25 (B) 0.40 (C) 0.45 (D) 0.56 (E) 0.65

   Answer: E: 0.65

   Hint/Solution: The probability breaks down into three mutually exclusive cases: (A1 minor, A2 minor), (A1 minor, A2 moderate), and (A1 moderate, A2 moderate), where A1 is the first accident, A2 the second. The probabilities for each these cases can be computed, using the independence assumption, by multiplying out the individual probabilities. Adding them up gives the result.

5. [Problem 1-5]

   An actuary studying the insurance preferences of automobile owners makes the following conclusions:

   (i) An automobile owner is twice as likely to purchase collision coverage as disability coverage.
   (ii) The event that an automobile owner purchases collision coverage is independent of the event that he or she purchases disability coverage.
   (iii) The probability that an automobile owner purchases both collision and disability coverages is 0.15.

   What is the probability that an automobile owner purchases neither collision nor disability coverage?

   (A) 0.18 (B) 0.33 (C) 0.48 (D) 0.67 (E) 0.82

   Answer: B: 0.33

   Hint/Solution: Let C denote collision coverage, and D disability coverage. We are given that (i) $P(C) = 2P(D)$, (ii) C and D are independent, and (iii) $P(C \cap D) = 0.15$, and we need to compute $P(C' \cap D')$.

   By independence, $P(C \cap D) = P(C)P(D)$. Using the independence along with the equations (i) and (iii) we get $0.15 = P(C \cap D) = P(C)P(D) = 2P(D)^2$, so $P(D) = 0.273$ and $P(C) = 0.548$. Hence $P(C') = 0.452$ and $P(D') = 0.727$. Applying independence again, we get $P(C' \cap D') = P(C')P(D') = 0.328$.

6. [Problem 1-6]

   An insurance company estimates that 40% of policyholders who have only an auto policy will renew next year and 60% of policyholders who have only a homeowners policy will renew next year. The company estimates that 80% of policyholders who have both an auto and a homeowners policy will renew at least one of those policies next year. Company records show that 65% of policyholders have an auto policy, 50% of policyholders have a homeowners policy, and 15% of policyholders have both an auto and a homeowners policy. Using the company's estimates, calculate the percentage of policyholders that will renew at least one policy next year.

   (A) 20 (B) 29 (C) 41 (D) 53 (E) 70

   Answer: D: 53

   Hint/Solution: The policyholders fall into 3 disjoint groups: only auto (A), only home (H), and auto and home (AH). From the given data we deduce $P(AH) = 0.15$, $P(A) = 0.65 - 0.15 = 0.5$, $P(H) = 0.5 - 0.15 = 0.35$, and (with R denoting renewal of policy) $P(R|A) = 0.4$, $P(R|H) = 0.6$, and $P(R|AH) = 0.8$. We want $P(R)$. This can be computed from the above probabilities using the total probability formula.

7. [Problem 1-7]

   A survey of a group's viewing habits over the last year revealed the following information:

   (i) 28% watched gymnastics.
   (ii) 29% watched baseball.
   (iii) 19% watched soccer.
   (iv) 14% watched gymnastics and baseball.
   (v) 12% watched baseball and soccer.
   (vi) 10% watched gymnastics and soccer.
   (vii) 8% watched all three sports.

   Calculate the percentage of the group that watched none of the three sports during the last year.

   (A) 24 (B) 36 (C) 41 (D) 52 (E) 60

   Answer: D: 0.52

   Hint/Solution: The probability we want is 1 minus the probability of the union of the 3 events in question, and the latter can be computed using the formula for probability of union of 3 sets.

8. [Problem 1-8]

   An insurance company pays hospital claims. The number of claims that include emergency room or operating room charges is 85% of the total number of claims. The number of claims that do not include emergency room charges is 25% of the total number of claims. The occurrence of emergency room charges is independent of the occurrence of operating room charges on hospital claims. Calculate the probability that a claim submitted to the insurance company includes operating room charges.

   (A) 0.10 (B) 0.20 (C) 0.25 (D) 0.40 (E) 0.80

   Answer: D: 0.40

   Hint/Solution: The relevant events are "operating room charges" and "emergency room charges"; we are given the probability of the union of these two events, and the probability of the complement of the second one, and we need to compute the probability of the first; for that, either use appropriate probability rules (e.g., the formula for the probability of a union), or determine the probability from a Venn diagram.

9. [Problem 1-9]

   A blood test indicates the presence of a particular disease 95% of the time when the disease is actually present. The same test indicates the presence of the disease 0.5% of the time when the disease is not present. One percent of the population actually has the disease. Calculate the probability that a person has the disease given that the test indicates the presence of the disease.

   (A) 0.324 (B) 0.657 (C) 0.945 (D) 0.950 (E) 0.995

   Answer: B: 0.657

   Hint/Solution: A routine application of Bayes' Rule.

10. [Problem 1-10]

    Ten percent of a company's life insurance policyholders are smokers. The rest are nonsmokers. For each nonsmoker, the probability of dying during the year is 0.01. For each smoker, the probability of dying during the year is 0.05. Given that a policyholder has died, what is the probability that the policyholder was a smoker?

    (A) 0.05 (B) 0.20 (C) 0.36 (D) 0.56 (E) 0.90

    Answer: C: 0.36

    Hint/Solution: An easy Bayes' Rule problem.

11. [Problem 1-11]

    An auto insurance company insures drivers of all ages. An actuary compiled the following statistics on the company's insured drivers:

Age of driver	Probability of accident	Portion of company's insured drivers
16–20	0.06	0.08
21–30	0.03	0.15
31–65	0.02	0.49
66–99	0.04	0.28

A randomly selected driver that the company insures has an accident. Calculate the probability that the driver was age 16-20.

(A) 0.13 (B) 0.16 (C) 0.19 (D) 0.23 (E) 0.40

Answer: B: 0.16

Hint/Solution: This is a routine application of Bayes' formula.

12. [Problem 1-12]

An actuary studied the likelihood that different types of drivers would be involved in at least one collision during any one-year period. The results of the study are presented below.

Type of driver	Percentage of all drivers	Probability of at least one collision
Teen	8%	0.15
Yound adult	16%	0.08
Midlife	45%	0.04
Senior	31%	0.05

Given that a driver has been involved in at least one collision in the past year, what is the probability that the driver is a young adult driver?

(A) 0.06 (B) 0.16 (C) 0.19 (D) 0.22 (E) 0.25

Answer: D: 0.22

Hint/Solution: A standard application of Bayes' Rule, but the calculations are somewhat tedious.

13. [Problem 1-13]

A health study tracked a group of persons for five years. At the beginning of the study, 20% were classified as heavy smokers, 30% as light smokers, and 50% as nonsmokers. Results of the study showed that light smokers were twice as likely as nonsmokers to die during the five-year study, but only half as likely as heavy smokers. A randomly selected participant from the study died over the five-year period. Calculate the probability that the participant was a heavy smoker.

(A) 0.20 (B) 0.25 (C) 0.35 (D) 0.42 (E) 0.57

Answer: D: 0.42

Hint/Solution: A standard exercise in Bayes' Rule; the only non-routine part here is to properly interpret the phrase "light smokers were twice as likely as nonsmokers, and half as likely as heavy smokers to die ...". This boils down to relations between the conditional probabilities of dying given a light smoker, a nonsmoker, and a heavy smoker.

14. [Problem 1-14]

The probability that a randomly chosen male has a circulation problem is 0.25. Males who have a circulation problem are twice as likely to be smokers as those who do not have a circulation problem. What is the conditional probability that a male has a circulation problem, given that he is a smoker?

(A) 1/4 (B) 1/3 (C) 2/5 (D) 1/2 (E) 2/3

Answer: C: 2/5

Hint/Solution: Not particularly difficult, but somewhat unconventional since the conditional probabilities are not explicitly given. Instead we are given that P(S—C) = 2 P(S—C'), S=smokes, C=has circulation problem. Substituting this into Bayes' Rule, the term P(S—C) will drop out.

15. [Problem 1-19]

A doctor is studying the relationship between blood pressure and heartbeat abnormalities in her patients. She tests a random sample of her patients and notes their blood pressures (high, low, or normal) and their heartbeats (regular or irregular). She finds that:

(i) 14% have high blood pressure.
(ii) 22% have low blood pressure.
(iii) 15% have an irregular heartbeat.
(iv) Of those with an irregular heartbeat, one-third have high blood pressure.
(v) Of those with normal blood pressure, one-eighth have an irregular heartbeat.

What portion of the patients selected have a regular heartbeat and low blood pressure?

(A) 2% (B) 5% (C) 8% (D) 9% (E) 20%

Answer: E: 20%

Hint/Solution: Put the given probabilities into a matrix with rows representing blood pressure (high, normal, low) and columns representing heartbeat (irregular, regular). Use the fact that the sum of all entries must be 100 percent, fill out the remaining entries, and read off the entry corresponding to low blood pressure and regular heartbeat.

16. [Problem 1-51]

An urn contains 10 balls, 4 red and 6 blue. A second urn contains 16 red balls and an unknown number of blue balls. A single ball is drawn from each urn. The probability that both balls are of the same color is 0.44. Calculate the number of blue balls in the second urn.

(A) 4 (B) 20 (C) 24 (D) 44 (E) 64

Answer: A: 4

Hint/Solution: The event "both balls have the same color", which we know has probability 0.44, splits into two disjoint events, "ball 1 red and ball 2 red" and "ball 1 blue and ball 2 blue". Since the drawings from the two urns are obviously independent, it follows that

$$0.44 = P(\text{ball 1 red}) \cdot P(\text{ball 2 red}) + P(\text{ball 1 red}) \cdot P(\text{ball 2 red}).$$

Since Urn 1 has 4 red balls and 6 blue balls, the probability for drawing a red ball from this urn is $4/10$, and the probability for drawing a blue ball is $6/10$. The corresponding probabilities for Urn 2 are $16/(16+b)$, and $b/(16+b)$, where b denotes the number of blue balls in Urn 2. Thus we get

$$0.44 = \frac{4}{10} \cdot \frac{16}{16+b} + \frac{6}{10} \cdot \frac{b}{16+b}.$$

Solving this equation for b gives $b = (0.44 - 0.4) \cdot 16/(0.6 - 0.44) = 4$.

17. [Problem 1-52]

A public health researcher examines the medical records of a group of 937 men who died in 1999 and discovers that 210 of the men died from causes related to heart disease. Moreover, 312 of the 937 men had at least one parent who suffered from heart disease, and, of these 312 men, 102 died from causes related to heart disease. Determine the probability that a man randomly selected from this group died of causes related to heart disease, given that neither of his parents suffered from heart disease.

(A) 0.115 (B) 0.173 (C) 0.224 (D) 0.327 (E) 0.514

Answer: B: 0.173

Hint/Solution: Let A denote the set of those in the group who died from causes related to heart disease, and B the set of those who had a parent with heart disease. We need to compute $P(A|B')$. From the given data we have

$$P(A) = \frac{210}{937} = 0.224, \quad P(B) = \frac{312}{937} = 0.333, \quad P(A \cap B) = \frac{102}{937} = 0.109.$$

Thus,

$$P(A|B') = \frac{P(A \cap B')}{P(B')} = \frac{P(A) - P(A \cap B)}{1 - P(B)} = \frac{0.224 - 0.109}{1 - 0.333} = 0.173.$$

18. [Problem 1-53]

Among a large group of patients recovering from shoulder injuries, it is found that 22% visit both a physical therapist and a chiropractor, whereas 12% visit neither of these. The probability that a patient visits a chiropractor exceeds by 0.14 the probability that a patient visits a physical therapist. Determine the probability that a randomly chosen member of this group visits a physical therapist.

(A) 0.26 (B) 0.38 (C) 0.40 (D) 0.48 (E) 0.62

Answer: D: 0.48

Hint/Solution: Let T denote the event "visits a physical therapist", and C the event "visits a chiropractor". We need to compute $P(T)$. We are given

$$P(T \cap C) = 0.22, \quad P(T' \cap C') = 0.12, \quad P(C) = P(T) + 0.14.$$

From the first and third equations we deduce

$$\begin{aligned} P(T \cup C) &= P(T) + P(C) - P(T \cap C) \\ &= P(T) + (P(T) + 0.14) - 0.22 = 2P(T) - 0.08. \end{aligned}$$

On the other hand, from the second equation and De Morgan's Law, we get

$$P(T \cup C) = 1 - P(T' \cap C') = 1 - 0.12 = 0.88.$$

Hence $2P(T) - 0.08 = 0.88$, so $P(T) = (0.88 + 0.08)/2 = 0.48$.

19. [Problem 1-54]

An insurer offers a health plan to the employees of a large company. As part of this plan, the individual employees may choose exactly two of the supplementary coverages A, B, and C, or they may choose no supplementary coverage. The proportions of the company's employees that choose coverages A, B, and C are $1/4$, $1/3$, and $5/12$, respectively. Determine the probability that a randomly chosen employee will choose no supplementary coverage.

(A) 0 (B) $\frac{47}{144}$ (C) $\frac{1}{2}$ (D) $\frac{97}{144}$ (E) $\frac{7}{9}$

Answer: C: 1/2

Hint/Solution: We need to find $P(A' \cap B' \cap C')$, where A, B, C denote the employees choosing coverages A, B, and C. By De Morgan's Law (or, alternatively, by a Venn diagram), this is equal to $1 - P(A \cup B \cup C)$. Thus, it remains to compute the latter probability.

The key now is to take into account the given assumption that each employee chooses exactly two of the three coverages, or none. This assumption implies that every employee that belongs to one of A, B, and C, i.e., every employee in the union $A \cup B \cup C$, belongs to exactly two of these sets, i.e., to exactly one of the intersections $A \cap B$, $A \cap C$, and $B \cap C$. Hence, the probability of the union $A \cup B \cup C$ is the sum of the probabilities of these three pairwise intersections, and it remains to compute these latter probabilities.

Since an employee choosing coverage A must choose one of coverages B and C, but not both, A is a union of $A \cap B$ and $A \cap C$, with the latter two sets disjoint. hence,

$$\frac{1}{4} = P(A) = P(A \cap B) + P(A \cap C).$$

Similarly,

$$\frac{1}{3} = P(B) = (A \cap B) + P(B \cap C),$$

and

$$\frac{5}{12} = P(C) = (A \cap C) + P(B \cap C).$$

Regarding these three equations as a system of equations in the three unknowns $P(A \cap B)$, $P(A \cap C)$ and $P(B \cap C)$ and solving, we get

$$P(A \cap B) = \frac{1}{12}, \quad P(A \cap C) = \frac{1}{6}, \quad P(B \cap C) = \frac{1}{4}.$$

Hence,

$$P(A' \cap B' \cap C') = 1 - \left(\frac{1}{12} + \frac{1}{6} + \frac{1}{4}\right) = \frac{1}{2}.$$

20. [Problem 1-55]

    An insurance company issues life insurance policies in three separate categories: standard, preferred, and ultra-preferred. Of the company's policyholders, 50% are standard, 40% are preferred, and 10% are ultra-preferred. Each standard policyholder has probability 0.010 of dying in the next year, each preferred policyholder has probability 0.005 of dying in the next year, and each ultra-preferred policyholder has probability 0.001 of dying in the next year. A policyholder dies in the next year. What is the probability that the deceased policyholder was ultra-preferred?

    (A) 0.0001 (B) 0.0010 (C) 0.0071 (D) 0.0141 (E) 0.2817

    Answer: D: 0.0141

    Hint/Solution: This is a standard Bayes' Rule exercise. Let S, P, U denote the standard, preferred, and ultra-preferred policyholders, and let D denote the event "dies in the next year". We need to compute $P(U|D)$.

    Applying Bayes' Rule with S, P, U as the partition of the sample space, and substituting the given data, we get

$$\begin{aligned} P(U|D) &= \frac{P(D|U)P(U)}{P(D|U)P(U) + P(D|S)P(S) + P(D|P)P(P)} \\ &= \frac{0.001 \cdot 0.1}{0.001 \cdot 0.1 + 0.01 \cdot 0.5 + 0.005 \cdot 0.4} = 0.01408. \end{aligned}$$

21. [Problem 1-56]

    Upon arrival at a hospital's emergency room, patients are categorized according to their condition as critical, serious, or stable. In the past year:

    (i) 10% of the emergency room patients were critical;

    (ii) 30% of the emergency room patients were serious;

    (iii) the rest of the emergency room patients were stable;

(iv) 40% of the critical patients died;

(vi) 10% of the serious patients died; and

(vii) 1% of the stable patients died.

Given that a patient survived, what is the probability that the patient was categorized as serious upon arrival?

(A) 0.06 (B) 0.29 (C) 0.30 (D) 0.39 (E) 0.64

Answer: B: 0.29

Hint/Solution: This is a standard Bayes' Rule exercise. Let C, S, T denote the events "critical", "serious", and "stable", and let D denote the event "patient dies". We need to compute $P(S|D')$.

Applying Bayes' Rule with C, S, T as the partition of the sample space, and substituting the given data, we get

$$\begin{aligned} P(S|D') &= \frac{P(D'|S)P(S)}{P(D'|S)P(S) + P(D'|C)P(C) + P(D'|T)P(T)} \\ &= \frac{(1-0.1)0.3}{(1-0.1)0.3 + (1-0.4)0.1 + (1-0.01)0.6} = 0.292. \end{aligned}$$

(Note the use here of the complement formula for conditional probabilities: $P(D'|C) = 1 - P(D|C) = 1 - 0.1$, etc.)

22. [Problem 1-101]

A large pool of adults earning their first driver's license includes 50% low-risk drivers, 30% moderate-risk drivers, and 20% high-risk drivers. Because these drivers have no prior driving record, an insurance company considers each driver to be randomly selected from the pool. This month, the insurance company writes 4 new policies for adults earning their first driver's license. What is the probability that these 4 will contain at least two more high-risk drivers than low-risk drivers?

(A) 0.006 (B) 0.012 (C) 0.018 (D) 0.049 (E) 0.073

Answer: D: 0.049

Hint/Solution: This requires a tedious case distinction, according to the types of the four drivers: Denoting high, moderate, and low risk drivers by the letters H, M, and L, we need to considers "words" of 4 of these letters containing at least two more H's than L's. The possible patterns and their probabilities are:

(a) 4 H's: HHHH: probability 0.2^4;

(b) 3 H's and 1 L: HHHL, HHLH, HLHH, LHHH: probability $0.2^3 0.5$ each, and $4 \cdot 0.2^3 0.5$ total;

(c) 3 H's and 1 M: HHHM, HHMH, HMHH, MHHH: probability $0.2^3 0.3$ each, and $4 \cdot 0.2^3 0.3$ total;

(d) 2 H's and 2 M: HHMM, HMHM, HMMH, MMHH, MHMH, MHHM; probability $0.2^3 0.3^2$ each, $6 \cdot 0.2^3 0.3^2$ total.

Adding up these probabilities, we get the answer:

$$0.2^4 + 4 \cdot 0.2^3 0.5 + 4 \cdot 0.2^3 0.3 + 6 \cdot 0.2^2 0.3^2 = 0.0489$$

23. [Problem 1-102]

A study of automobile accidents produced the following data:

Model year	Proportion of all vehicles	Probability of involvement in an accident
1997	0.16	0.05
1998	0.18	0.02
1999	0.20	0.03
Other	0.46	0.04

An automobile from one of the model years 1997, 1998, and 1999 was involved in an accident. Determine the probability that the model year of this automobile is 1997.

(A) 0.22 (B) 0.30 (C) 0.33 (D) 0.45 (E) 0.50

Answer: D: 0.45

Hint/Solution: The tricky part is the restriction to model years 1997, 1998, and 1999; to take this into account, simply consider these three years as the entire universe (rescaling the proportions for these years so that they add up to 1). Once this is done, the problem reduces to a standard application of Bayes' Rule.

24. [Problem 1-103]

An insurance company examines its pool of auto insurance customers and gathers the following information:

(i) All customers insure at least one car.
(ii) 70% of the customers insure more than one car.
(iii) 20% of the customers insure a sports car.
(iv) Of those customers who insure more than one car, 15% insure a sports car.

Calculate the probability that a randomly selected customer insures exactly one car and that car is not a sports car.

(A) 0.13 (B) 0.21 (C) 0.24 (D) 0.25 (E) 0.30

Answer: B: 0.21

Hint/Solution: The tricky part here is the proper interpretation of the given data and the question asked. Letting A denote the event "insures more than one car" and B the event "insures a sports car", we need to compute $P(A' \cap B')$. (Note that the complement to A, "at most one car", is equivalent to "exactly one car", by the assumption that all customers insure at least one car.)

In terms of this notation, the given data translates into $P(A) = 0.7$, $P(B) = 0.2$, $P(B|A) = 0.15$, and from this we deduce $P(A \cap B) = P(B|A)P(A) = 0.105$. We now have everything at hand to compute the requested probability:

$$\begin{aligned} P(A' \cap B') &= 1 - P(A \cup B) = 1 - P(A) - P(B) + P(A \cap B) \\ &= 1 - 0.7 - 0.2 + 0.105 = 0.205. \end{aligned}$$

25. [Problem 1-104]

An auto insurance company has 10,000 policyholders. Each policyholder is classified as

(i) young or old;
(ii) male or female; and
(iii) married or single.

Of these policyholders, 3000 are young, 4600 are male, and 7000 are married. The policyholders can also be classified as 1320 young males, 3010 married males, and 1400 young married persons. Finally, 600 of the policyholders are young married males. How many of the company's policyholders are young, female, and single?

(A) 280 (B) 423 (C) 486 (D) 880 (E) 896

Answer: D: 880

Hint/Solution: If we consider the sets "young", "male" and "married", we want to count the part of the "young" set that is outside the other two sets. Drawing a Venn diagram, we see that this count is given by

$$\begin{aligned} &\#\{\text{young}\} \quad - \quad \#\{\text{young and married}\} \quad - \quad \#\{\text{young and male}\} \\ &\qquad + \quad \#\{\text{young and married and male}\} \\ &= 3000 - 1320 - 1400 + 600 = 880. \end{aligned}$$

26. [Problem 1-105]

A dental insurance policy covers three procedures: orthodontics, fillings, and extractions. During the life of the policy, the probability that the policyholder needs

- orthodontic work is 1/2;
- orthodontic work or a filling is 2/3;

- orthodontic work or an extraction is 3/4,
- a filling and an extraction is 1/8.

The need for orthodontic work is independent of the need for a filling and is independent of the need for an extraction. Calculate the probability that the policyholder will need a filling or an extraction during the life of the policy.

(A) 7/24 (B) 3/8 (C) 2/3 (D) 17/24 (E) 5/6

Answer: D: 17/24

Hint/Solution: Let O, F, E denote the three procedures. Then we need to compute $P(E \cup F)$. From the given data, (i) $P(O) = 1/2$, (ii) $P(O \cup F) = 2/3$, (iii) $P(O \cup E) = 3/4$, (iv) $P(F \cap E) = 1/3$. We also know that O and F are independent, and that O and E are independent (but not E and F).

From (i), (ii), and the independence of O and F we can determine $P(F)$:

$$\frac{2}{3} = P(O \cup F) = P(O) + P(F) - P(O \cap F)$$
$$= P(O) + P(F) - P(O)P(F) = \frac{1}{2} + P(F) - \frac{1}{2}P(F).$$

Solving for $P(F)$ gives $P(F) = 1/3$. Similarly, from (i) and (iii) and the independence of O and E we get $P(E) = 1/2$. Hence,

$$P(E \cup F) = P(E) + P(F) - P(E \cap F) = \frac{1}{2} + \frac{1}{3} - \frac{1}{8} = \frac{17}{24}.$$

27. [Problem 1-106]

An actuary is studying the prevalence of three health risk factors, denoted by A, B, and C, within a population of women. For each of the three factors, the probability is 0.1 that a woman in the population has only this risk factor (and no others). For any two of the three factors, the probability is 0.12 that she has exactly these two risk factors (but not the other). The probability that a woman has all three risk factors, given that she has A and B, is 1/3. What is the probability that a woman has none of the three risk factors, given that she does not have risk factor A?

(A) 0.280 (B) 0.311 (C) 0.467 (D) 0.484 (E) 0.700

Answer: C: 0.467

Hint/Solution: This is a lengthy set-theoretic exercise involving three intersecting sets. The probability to compute is $P(A' \cap B' \cap C' | A') = P(A' \cap B' \cap C')/P(A')$, so we need to compute $P(A' \cap B' \cap C')$ and $P(A')$. Now, $P(A') = 1 - P(A)$ and by De Morgan's Law, $P(A' \cap B' \cap C') = 1 - P(A \cup B \cup C)$. Thus, the calculation reduces to that of $P(A \cup B \cup C)$ and $P(A)$.

This is best done with the aid of a Venn diagram. The given probabilities 0.1 and 0.12 represent "areas" in this Venn diagram occupied by exactly one, respectively exactly two, of the three sets A, B, C. The only unknown in this diagram is the area of the triple intersection, $P(A \cap B \cap C)$, and determining this area is the trickiest part of the problem. To this end, we exploit the final bit of information given in the problem, which says that the conditional probability $P(A \cap B \cap C|A \cap B)$ is 1/3. This translates into $P(A \cap B \cap C)/P(A \cap B) = 1/3$. Thus the triple intersection occupies 1/3 of the double intersection of A and B. Now, from the Venn diagram and the given information we see that the remaining part (i.e., the remaining 2/3) of this double intersection has area 0.12. Hence the entire intersection of $A \cap B$ must have area 0.18, and the triple intersection $A \cap B \cap C$ has area 0.06. Adding up the appropriate areas, we see that that $P(A) = 0.1 + 2 \cdot 0.12 + 0.06 = 0.4$, $P(A \cup B \cup C) = 3 \cdot 0.1 + 3 \cdot 0.12 + 0.06 = 0.72$. Hence the probability sought is $(1 - 0.72)/(1 - 0.4) = 7/15 = 0.467$.

28. [Problem 1-107]

An insurance company designates 10% of its customers as high risk and 90% as low risk. The number of claims made by a customer in a calendar year is Poisson distributed with mean θ and is independent of the number of claims made by a customer in the previous calendar year. For high risk customers, $\theta = 0.6$, while for low risk customers $\theta = 0.1$. Calculate the expected number of claims made in calendar year 1998 by a customer who made one claim in calendar year 1997.

(A) 0.15 (B) 0.18 (C) 0.24 (D) 0.30 (E) 0.40

Answer: C: 0.24

Hint/Solution: A devilish problem—one of the most difficult actuarial exam problems I have seen! What makes this problem so tricky is the restriction to customers who made one claim in 1997 in the expectation to be computed. Without this restriction the expected number of claims would simply be an average of the expected numbers for high and low risk customers, weighted by the probabilities of high and low risk customers, i.e., $0.6 \cdot 0.1 + 0.1 \cdot 0.9 = 0.15$.

To take the above restriction into account, one has to replace the weights 0.1 and 0.9 in this calculation by corresponding *conditional* probabilities, the condition being that the customer had one accident in 1997. More formally, let H and L denote high and low risk customers, respectively, and let A denote the event "one claim made in 1997". Then the above weights are $P(H)(= 0.1)$ and $P(L)(= 0.9)$, and we need to replace these by $P(H|A)$ and $P(L|A)$, respectively.

The computation of these conditional probabilities is a nontrivial exercise in itself. By Bayes' rule we have

$$P(H|A) = \frac{P(A|H)P(H)}{P(A|H)P(H) + P(A|L)P(L)}.$$

By definition, $P(A|H)$ is the probability that (exactly) one claim is made, *given that the customer is high risk*. Since for high risk drivers the number of claims is Poisson distributed with mean 0.6, this probability is equal to $e^{-0.6}0.6$. Analogously, we get $P(A|L) = e^{-0.1}0.1$. Substituting these values along with $P(H) = 0.1$ and

$P(L) = 0.9$ into the above formula, we get $P(H|A) = 0.288$, and hence $P(L|A) = 1 - P(H|A) = 0.712$, by the complement formula for conditional probabilities. Using these weights in place of 0.1 and 0.9 in the above calculation, we get the correct answer: $0.6 \cdot 0.288 + 0.1 \cdot 0.712 = 0.244$.