Math 370, Spring 2008
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Practice Test 1 Solutions

About this test. This is a practice test made up of a random collection of 15 problems from past Course 1/P actuarial exams. Most of the problems have appeared on the Actuarial Problem sets passed out in class, but I have also included a number of additional problems.

Topics covered. This test covers the topics of Chapters 1–3 in Hogg/Tanis and Actuarial Problem Sets 1–3: General probability, discrete distributions, and continuous distributions.

Ordering of the problems. In order to mimick the conditions of the actual exam as closely as possible, the problems are in no particular order. Easy problems are mixed in with hard ones. In fact, I used a program to select the problems and to put them in random order, with no human intervention. If you find the problems hard, it's the luck of the draw!

Suggestions on taking the test. Try to take this test as if it were the real thing. Take it as a closed books, notes, etc., time yourself, and stop after 1:30 hours. In the actuarial exam you have 3 hours for 30 problems, so 1:30 is an appropriate time limit for a 15 problem test.

Answers/solutions. Answers and solutions will be posted on the course webpage, `www.math.uiuc.edu/~hildebr/370`.

1. [3-22]

   An actuary is reviewing a study she performed on the size of the claims made ten years ago under homeowners insurance policies. In her study, she concluded that the size of claims followed an exponential distribution and that the probability that a claim would be less than $1,000 was 0.250. The actuary feels that the conclusions she reached in her study are still valid today with one exception: every claim made today would be twice the size of a similar claim made ten years ago as a result of inflation. Calculate the probability that the size of a claim made today is less than $1,000.

   Answer: C: 0.134

   Solution: First determine the mean θ from the equation $F(1000) = 0.25$, then compute $F(500)$ (note that 500 is the claim size today corresponding to a claim size of 1000 ten years ago).

2. [2-104]

   A study is being conducted in which the health of two independent groups of ten policyholders is being monitored over a one-year period of time. Individual participants in the study drop out before the end of the study with probability 0.2 (independently of the other participants). What is the probability that at least 9 participants complete the study in one of the two groups, but not in both groups?

   (A) 0.096 (B) 0.192 (C) 0.235 (D) 0.376 (E) 0.469

   Answer: E: 0.469

   Solution: This is a two stage problem. For the first stage, focus on a single group, and consider the probability, say q, that in this group at least 9 participants complete the study. Then q is the probability that in 10 success/failure trials with success probability $p = 0.8$ there are at least 9 successes, so q can be computed using the binomial distribution: $q = 1 - 0.8^{10} - \binom{10}{9}0.8^9 0.2 = 0.62419$.

   For the second stage, we consider two such groups, and compute the probability that exactly one of them is "successful" in the above sense. This probability is given by $2q(1-q) = 0.46939$, as can be seen either by considering separately the cases "group 1 successful, group 2 unsuccessful" and "group 1 unsuccessful, group 2 successful", or by another application of the success/trial model, this time with 2 trials and success probability q.

3. [1-12]

   An actuary studied the likelihood that different types of drivers would be involved in at least one collision during any one-year period. The results of the study are presented below.

Type of driver	Percentage of all drivers	Probability of at least one collision
Teen	8%	0.15
Young adult	16%	0.08
Midlife	45%	0.04
Senior	31%	0.05

Given that a driver has been involved in at least one collision in the past year, what is the probability that the driver is a young adult driver?

(A) 0.06 (B) 0.16 (C) 0.19 (D) 0.22 (E) 0.25

Answer: D: 0.22

Solution: This is a standard Bayes' Rule exercise. Let T, Y, M, S denote the four types of drivers (teen, young adult, midlife, senior), and let C denote the event that the driver is involved in at least one collision. We need to compute $P(Y|C)$.

Applying Bayes' Rule with T, Y, M, S as the partition of the sample space, and substituting the given data, we get

$$\begin{aligned} P(Y|C) &= \frac{P(C|Y)P(Y)}{P(C|T)P(T) + P(C|Y)P(Y) + P(C|M)P(M) + P(C|S)P(S)} \\ &= \frac{0.08 \cdot 0.16}{0.15 \cdot 0.08 + 0.08 \cdot 0.16 + 0.04 \cdot 0.45 + 0.05 \cdot 0.31} = 0.22. \end{aligned}$$

4. [2-52]

An insurance company determines that N, the number of claims received in a week, is a random variable with $P(N = n) = 2^{-n-1}$, where $n \geq 0$. The company also determines that the number of claims received in a given week is independent of the number of claims received in any other week. Determine the probability that exactly seven claims will be received during a given two-week period.

(A) $\frac{1}{256}$ (B) $\frac{1}{128}$ (C) $\frac{7}{512}$ (D) $\frac{1}{64}$ (E) $\frac{1}{32}$

Answer: D: 1/64

Solution: Let N_1 and N_2, denote the number of claims during the first, respectively second, week. The total number of claims received during the two-week period then is $N_1 + N_2$, and so we need to compute $P(N_1 + N_2 = 7)$. By breaking this probability down into cases according to the values of (N_1, N_2), and using the independence of N_1 and N_2 and the given distribution, we get

$$\begin{aligned} P(N_1 + N_2 = 7) &= \sum_{n=0}^{7} P(N_1 = n, N_2 = 7 - n) = \sum_{n=0}^{7} 2^{-n-1} 2^{-(7-n)-1} \\ &= \sum_{n=0}^{7} 2^{-9} = 8 \cdot 2^{-9} = \frac{1}{64}. \end{aligned}$$

5. [2-1]

An insurance policy pays an individual 100 per day for up to 3 days of hospitalization and 25 per day for each day of hospitalization thereafter. The number of days of hospitalization, X, is a discrete random variable with probability function

$$P(X = k) = \begin{cases} \frac{6-k}{15} & \text{for } k = 1, 2, 3, 4, 5, \\ 0 & \text{otherwise.} \end{cases}$$

Calculate the expected payment for hospitalization under this policy.

(A) 85 (B) 163 (C) 168 (D) 213 (E) 255

Answer: D: 213

Solution: Letting X denote the number of days in the hospital, and Y the total payment, the given information can be summarized in the following table:

k	1	2	3	4	5
$P(X=k)$	$\frac{5}{15}$	$\frac{4}{15}$	$\frac{3}{15}$	$\frac{2}{15}$	$\frac{1}{15}$
Y	100	200	300	325	350

Hence, the expected payment is

$$E(Y) = 100 \cdot \frac{5}{15} + 200 \cdot \frac{4}{15} + 300 \cdot \frac{3}{15} + 325 \cdot \frac{2}{15} + 350 \cdot \frac{1}{15} = 213.3$$

6. [2-3]

An actuary has discovered that policyholders are three times as likely to file two claims as to file four claims. If the number of claims filed has a Poisson distribution, what is the variance of the number of claims filed?

(A) $1/\sqrt{3}$ (B) 1 (C) $\sqrt{2}$ (D) 2 (E) 4

Answer: D: 2

Solution: Let X denote the number of claims. We are given that X has Poisson distribution and that $P(X=2) = 3P(X=4)$. Substituting the formula for a Poisson p.m.f., $P(X=x) = e^{-\lambda}\lambda^x/x!$ into this equation, we get $e^{-\lambda}\lambda^2/2! = 3e^{-\lambda}\lambda^4/4!$, which implies $\lambda^2 = 4$, and therefore $\lambda = 2$ since λ must be positive. Since the variance of a Poisson distribution is given by $\sigma^2 = \lambda$, it follows that $\sigma^2 = 2$.

7. [3-20]

A device that continuously measures and records seismic activity is placed in a remote region. The time, T, to failure of this device is exponentially distributed with mean 3 years. Since the device will not be monitored during its first two years of service, the time to discovery of its failure is $X = \max(T, 2)$. Determine $E(X)$.

(A) $2 + \frac{1}{3}e^{-6}$
(B) $2 - 2e^{-2/3} + 5e^{-4/3}$
(C) 3
(D) $2 + 3e^{-2/3}$
(E) 5

Answer: D

Solution: Since T is exponentially distributed with mean 3, the density of T is $f(t) = (1/3)e^{-t/3}$ for $t > 0$. Since $X = \max(T, 2)$, we have $X = 2$ if $0 \le T \le 2$ and $X = T$ if $2 < T < \infty$.

Thus,

$$\begin{aligned} E(X) &= \int_0^2 2\frac{1}{3}e^{-t/3}dt + \int_2^\infty t \cdot \frac{1}{3}e^{-t/3}dt \\ &= 2(1 - e^{-2/3}) - te^{-t/3}\Big|_2^\infty + \int_2^\infty e^{-t/3}dt \\ &= 2(1 - e^{-2/3}) + 2e^{-t/3} + 3e^{-2/3} = 2 + 3e^{-2/3} \end{aligned}$$

8. [3-56]

A tour operator has a bus that can accommodate 20 tourists. The operator knows that tourists may not show up, so he sells 21 tickets. The probability that an individual tourist will not show up is 0.02, independent of all other tourists. Each ticket costs 50, and is non-refundable if a tourist fails to show up. If a tourist shows up and a seat is not available, the tour operator has to pay 100 (ticket cost + 50 penalty) to the tourist. What is the expected revenue of the tour operator?

(A) 935 (B) 950 (C) 967 (D) 976 (E) 985

Answer: E:985

Solution: This is simpler than it might seem since there are only two cases to consider:

(i) All 21 ticket holders show up. This occurs with probability $(1-0.02)^{21} = 0.654$. In this case, the operator earns $21 \cdot 50 = 1050$ from ticket sales, but incurs a cost of 100 for the one ticket holder that cannot be accommodated, so the revenue is $1050 - 100 = 950$.

(ii) Not all 21 ticket holders show up. This occurs with the complementary probability, $1 - 0.654 = 0.346$. In this case, the operator earns 1050 from ticket sales, but incurs no penalty, so the revenue is 1050.

Hence the expected revenue is $950 \cdot 0.654 + 1050 \cdot 0.346 = 985$.

9. [3-14]

Under a group insurance policy, an insurer agrees to pay 100% of the medical bills incurred during the year by employees of a small company, up to a maximum total of one million dollars. The total amount of bills incurred, X, has probability density function

$$f(x) = \begin{cases} \frac{x(4-x)}{9} & \text{for } 0 < x < 3, \\ 0 & \text{otherwise,} \end{cases}$$

where x is measured in millions. Calculate the total amount, in millions of dollars, the insurer would expect to pay under this policy.

(A) 0.120 (B) 0.301 (C) 0.935 (D) 2.338 (E) 3.495

Answer: C: 0.935

Solution: Let Y denote the payout (in millions). Then $Y = X$ if $X \leq 1$, and $Y = 1$ otherwise. Thus,

$$\begin{aligned} E(Y) &= \int_0^1 xf(x)dx + \int_1^\infty 1 \cdot f(x)dx \\ &= \int_0^1 \frac{1}{9}x^2(4-x)dx + \int_1^3 \frac{1}{9}x(4-x)dx \\ &= \frac{1}{9}\left(4\frac{1}{3} - \frac{1}{4}\right) + \frac{1}{9}\left(\frac{3^2-1^2}{2} - \frac{3^3-1^3}{3}\right) \\ &= \frac{13}{108} + \frac{22}{27} = 0.935. \end{aligned}$$

10. [3-4]

The loss amount, X, for a medical insurance policy has cumulative distribution function

$$F(x) = \begin{cases} 0, & x < 0, \\ \frac{1}{9}\left(2x^2 - \frac{x^3}{3}\right), & 0 \le x \le 3, \\ 1, & x > 3. \end{cases}$$

Calculate the mode of the distribution.

(A) $\frac{2}{3}$ (B) 1 (C) $\frac{3}{2}$ (D) 2 (E) 3

Answer: D: 2

Solution: The mode of a distribution is the point x at which $f(x)$ is maximal. With the given distribution we have $f(x) = F'(x) = (4/9)x - (1/9)x^2$ for $0 < x < 3$, and $f(x) = 0$ outside the interval $[0, 3]$. Differentiating, we get $f'(x) = (4/9) - (2/9)x$, and setting this equal to 0, we see that $x = 2$ is the only critical point of f. Hence the maximum must occur at $x = 2$.

11. [3-2]

Let X be a continuous random variable with density function

$$f(x) = \begin{cases} \frac{|x|}{10} & \text{for } -2 \le x \le 4, \\ 0 & \text{otherwise.} \end{cases}$$

Calculate the expected value of X.

(A) $\frac{1}{5}$ (B) $\frac{3}{5}$ (C) 1 (D) $\frac{28}{15}$ (E) $\frac{12}{5}$

Answer: D: 28/15

Solution: This is an easy integration exercise. The only tricky part here is the absolute value $|x|$ appearing in the integrand. This requires splitting the integral into two parts, corresponding to ranges $-2 \le x \le 0$ and $0 < x \le 4$, replacing $|x|$ by $-x$ in the first range, and by x in the second range:

$$\begin{aligned} E(X) &= \int_{-2}^{4} x f(x) dx = \int_{-2}^{4} x \frac{|x|}{10} dx \\ &= \frac{1}{10}\int_{-2}^{0} x(-x) dx + \frac{1}{10}\int_{0}^{4} x(x) dx \\ &= \frac{1}{10}\left(-\frac{2^3}{3}\right) + \frac{1}{10}\left(\frac{4^3}{3}\right) = \frac{28}{15}. \end{aligned}$$

12. [3-21]

An insurance policy pays for a random loss X subject to a deductible of C, where $0 < C < 1$. The loss amount is modeled as a continuous random variable with density function

$$f(x) = \begin{cases} 2x & \text{for } 0 < x < 1, \\ 0 & \text{otherwise.} \end{cases}$$

Given a random loss X, the probability that the insurance payment is less than 0.5 is equal to 0.64. Calculate C.

(A) 0.1 (B) 0.3 (C) 0.4 (D) 0.6 (E) 0.8

Answer: B: 0.3

Solution: The insurance payment, Y, is given by $Y = X - C$ if $X \geq C$, and $Y = 0$ if $X < C$. Thus, $Y \leq 0.5$ is equivalent to $X \leq 0.5 + C$, and we have

$$P(Y \leq 0.5) = P(X \leq C + 0.5) = \int_0^{C+0.5} 2x dx = (C + 0.5)^2.$$

Setting this equal to 0.64 and solving for C we get $(C+0.5)^2 = 0.64$, so $C = \sqrt{0.64} - 0.5 = 0.3$. (Note that this argument would not be correct if C were greater than 0.5 since then the upper limit in the integral falls outside the range of the density. However, it is easy to see that this case cannot occur: Since the given density function restricts X to the interval $0 \leq X \leq 1$, if $C > 1/2$, then $P(X \leq 0.5 + C) = 1$, and hence also $P(Y \leq 0.5) = 1$, contrary to the given condition $P(Y \leq 0.5) = 0.64$.

13. [3-16]

An insurance policy is written to cover a loss, X, where X has a uniform distribution on $[0, 1000]$. At what level must a deductible be set in order for the expected payment to be 25% of what it would be with no deductible?

(A) 250 (B) 375 (C) 500 (D) 625 (E) 750

Answer: C: 500

Solution: First note, that without the deductible the expected payment would be 500 (since it is then equal to the loss, and the loss X is uniformly distributed on $[0, 1000]$). Next, let D denote the (unknown) deductible and Y the payment under deductible D. Then $Y = X - D$ if $D \leq X \leq 1000$, and $Y = 0$ if $X \leq D$. Thus,

$$E(Y) = \int_D^{1000} (x-D) f(x) dx = \int_D^{1000} (x-D) \frac{1}{1000} dx = \frac{1}{1000} \cdot \frac{(1000 - D)^2}{2} = \frac{(1000 - D)^2}{2000}$$

Now set this expression equal to 25% of 500, i.e., 125, and solve for D: $(1000 - D)^2/2000 = 125$, or $(1000 - D)^2 = 250000$, so $D = 500$.

14. [2-70]

An insurance policyholder can submit up to 5 claims. The probability that the policyholder submits exactly n claims is p_n, for $n = 0, 1, 2, 3, 4, 5$. The following information is known:

(i) The difference between p_n and p_{n+1} is constant for $n = 0, 1, 2, 3, 4$.
(ii) Exactly 40% of the policyholders submit fewer than two claims.

What is the probability that a policholder submits more than three claims?

(A) 0.06 (B) 0.19 (C) 0.26 (D) 0.34 (E) 0.45

Answer: C: 0.26

Solution: The first condition on p_n implies that $p_{n+1} = p_n + c$ for $n = 0, 1, 2, 3, 4$ and some constant c. Hence $p_1 = p_0 + c$, $p_2 = p_1 + c = p_0 + 2c$, and in general,

$$p_n = p_0 + nc, \quad n = 0, 1, 2, 3, 4, 5.$$

Next, since the numbers p_n, $n = 0, 1, \dots, 5$, form a probability distribution, their sum must be 1, so we get the equation

$$1 = \sum_{n=0}^{5} (p_0 + nc) = 6p_0 + 15c.$$

The second condition translates into

$$0.4 = p_0 + p_1 = 2p_0 + c.$$

Solving the last two equations for p_0 and c, we obtain $c = -1/60$ and $p_0 = (0.4 - c)/2 = 1/5 + 1/120 = 5/24$. Hence the probability that a policy holder submits more than three claims is

$$p_4 + p_5 = 2p_0 + 9c = 2\frac{5}{24} + 9\frac{-1}{60} = \frac{4}{15} = 0.26.$$

15. [1-70]

A company offers a health insurance plan, a life insurance plan, and an investment insurance plan. An employee can have 0, 1, or 2 plans, but cannot have both life insurance and investment plans. You are given the following information:

(i) 450 employees have at least one plan.
(ii) 330 employees have only one plan.
(iii) 320 employees have the health insurance plan.
(iv) 45 employees have only the life insurance plan.
(v) There are 20 more employees that have both health and life plans than those that have both health and investment plans.

How many people have the investment plan?

(A) 85 (B) 120 (C) 135 (D) 155 (E) 180

Answer: C:135

Solution: The given conditions imply that, the 450 employees that have at least one plan fall into the following pairwise disjoint groups:

(i) Only health.
(ii) Only life.
(iii) Only investment.
(iv) Health and investment.
(v) Health and life.

Denote the number of employees in these five groups by H, L, I, HI, and HL, respectivley. Then

$$\begin{aligned} H + L + I + HI + HL &= 450, \\ H + L + I &= 330, \\ H + HI + HL &= 320, \\ L &= 45, \\ HL &= HI + 20 \end{aligned}$$

We solve these five equations for the five variables H, I, L, HI, HL:

$$\begin{aligned} HL + HI &= 450 - 330 = 120, \\ (HI + 20) + HI &= 120, \\ HI &= (120 - 20)/2 = 50, \\ HL &= 120 - HI = 120 - 50 = 70, \\ H &= 320 - 50 - 70 = 200, \\ I &= 330 - 200 - 45 = 85, \end{aligned}$$

Finally, the number of those that have the investment plan is

$$I + HI = 85 + 50 = 135.$$