Variance, covariance, and moment-generating functions

Practice problems — Solutions

1. Suppose that the cost of maintaining a car is given by a random variable, \( X \), with mean 200 and variance 260. If a tax of 20% is introduced on all items associated with the maintenance of the car, what will the variance of the cost of maintaining a car be?

Solution: The new cost is \( 1.2X \), so its variance is \( \text{Var}(1.2X) = 1.2^2 \text{Var}(X) = 1.44 \cdot 260 = 374 \).

2. The profit for a new product is given by \( Z = 3X - Y - 5 \), where \( X \) and \( Y \) are independent random variables with \( \text{Var}(X) = 1 \) and \( \text{Var}(Y) = 2 \). What is the variance of \( Z \)?

Solution: Using the properties of a variance, and independence, we get

\[
\text{Var}(Z) = \text{Var}(3X - Y - 5) = \text{Var}(3X - Y) = 9 \text{Var}(X) + \text{Var}(Y) = 11.
\]

3. An insurance policy pays a total medical benefit consisting of a part paid to the surgeon, \( X \), and a part paid to the hospital, \( Y \), so that the total benefit is \( X + Y \). Suppose that \( \text{Var}(X) = 5,000 \), \( \text{Var}(Y) = 10,000 \), and \( \text{Var}(X + Y) = 17,000 \). If \( X \) is increased by a flat amount of 100, and \( Y \) is increased by 10%, what is the variance of the total benefit after these increases?

Solution: We need to compute \( \text{Var}(X + 100 + 1.1Y) \). Since adding constants does not change the variance, this is the same as \( \text{Var}(X + 1.1Y) \), which expands as follows:

\[
\text{Var}(X + 1.1Y) = \text{Var}(X) + \text{Var}(1.1Y) + 2 \text{Cov}(X, 1.1Y) = \text{Var}(X) + 1.1^2 \text{Var}(Y) + 2 \cdot 1.1 \text{Cov}(X, Y).
\]

We are given that \( \text{Var}(X) = 5,000 \), \( \text{Var}(Y) = 10,000 \), so the only remaining unknown quantity is \( \text{Cov}(X, Y) \), which can be computed via the general formula for \( \text{Var}(X + Y) \):

\[
\text{Cov}(X, Y) = \frac{1}{2} (\text{Var}(X + Y) - \text{Var}(X) - \text{Var}(Y)) = \frac{1}{2} (17,000 - 5,000 - 10,000) = 1,000.
\]

Substituting this into the above formula, we get the answer:

\[
\text{Var}(X + 1.1Y) = 5,000 + 1.1^2 \cdot 10,000 + 2 \cdot 1.1 \cdot 1,000 = 19,520
\]

4. A company insure homes in three cities, J, K, L. The losses occurring in these cities are independent. The moment-generating functions for the loss distributions of the cities are

\[
M_J(t) = (1 - 2t)^{-3}, \quad M_K(t) = (1 - 2t)^{-2.5}, \quad M_L(t) = (1 - 2t)^{-4.5}
\]

Let \( X \) represent the combined losses from the three cities. Calculate \( E(X^3) \).

Solution: Let \( J, K, L \) denote the losses from the three cities. Then \( X = J + K + L \).
Since $J, K, L$ are independent, the moment-generating function for their sum, $X$, is equal to the product of the individual moment-generating functions, i.e.,

$$M_X(t) = M_K(t)M_J(t)M_L(t) = (1 - 2t)^{-3} - 2.5 - 4.5 = (1 - 2t)^{-10}.$$ 

Differentiating this function, we get

$$M'(t) = (-2)(-10)(1 - 2t)^{-11},$$

$$M''(t) = (-2)^2(-10)(-11)(1 - 2t)^{-12},$$

$$M'''(t) = (-2)^3(-10)(-11)(-12)(1 - 2t)^{-13}.$$ 

Hence, $E(X^3) = M'''(0) = (-2)^3(-10)(-11)(-12) = 10,560.$

5. Given that $E(X) = 5, E(X^2) = 27.4, E(Y) = 7, E(Y^2) = 51.4$ and $\text{Var}(X + Y) = 8$, find $\text{Cov}(X + Y, X + 1.2Y)$.

**Solution:** By definition,

$$\text{Cov}(X + Y, X + 1.2Y) = E((X + Y)(X + 1.2Y)) - E(X + Y)E(X + 1.2Y).$$

Using the properties of expectation and the given data, we get

\[
E((X + Y)(X + 1.2Y)) = E(X^2) + 2E(XY) + 1.2E(Y^2) \\
= 27.4 + 2E(XY) + 1.2 \cdot 51.4 = 2E(XY) + 89.08,
\]

$$\text{Cov}(X + Y, X + 1.2Y) = 2E(XY) + 89.08 - 160.8 = 2E(XY) - 71.72$$

To complete the calculation, it remains to find $E(XY)$. To this end we make use of the still unused relation $\text{Var}(X + Y) = 8$:

\[
8 = \text{Var}(X + Y) = E((X + Y)^2) - (E(X + Y))^2 = E(X^2) + 2E(XY) + E(Y^2) - (E(X) + E(Y))^2 \\
= 27.4 + 2E(XY) + 51.4 - (5 + 7)^2 = 2E(XY) - 65.2,
\]

so $E(XY) = 36.6$. Substituting this above gives $\text{Cov}(X + Y, X + 1.2Y) = 2.2 \cdot 36.6 - 71.72 = 8.8.$