Variance, covariance, and moment-generating functions

Practice problems — Solutions

1. Suppose that the cost of maintaining a car is given by a random variable, X, with mean 200 and variance 260. If a tax of 20% is introducted on all items associated with the maintenance of the car, what will the variance of the cost of maintaining a car be?

   Solution: The new cost is $1.2X$, so its variance is $\mathrm{Var}(1.2X) = 1.2^2\,\mathrm{Var}(X) = 1.44 \cdot 260 = 374$.

2. The profit for a new product is given by $Z = 3X - Y - 5$, where X and Y are independent random variables with $\mathrm{Var}(X) = 1$ and $\mathrm{Var}(Y) = 2$. What is the variance of Z?

   Solution: Using the properties of a variance, and independence, we get

   $$\mathrm{Var}(Z) = \mathrm{Var}(3X-Y-5) = \mathrm{Var}(3X-Y) = \mathrm{Var}(3X)+\mathrm{Var}(-Y) = 9\,\mathrm{Var}(X)+\mathrm{Var}(Y) = 11.$$

3. An insurance policy pays a total medical benefit consisting of a part paid to the surgeon, X, and a part paid to the hospital, Y, so that the total benefit is $X + Y$. Suppose that $\mathrm{Var}(X) = 5,000$, $\mathrm{Var}(Y) = 10,000$, and $\mathrm{Var}(X + Y) = 17,000$.

   If X is increased by a flat amount of 100, and Y is increased by 10%, what is the variance of the total benefit after these increases?

   Solution: We need to compute $\mathrm{Var}(X + 100 + 1.1Y)$. Since adding constants does not change the variance, this is the same as $\mathrm{Var}(X + 1.1Y)$, which expands as follows:

   $$\mathrm{Var}(X + 1.1Y) = \mathrm{Var}(X) + \mathrm{Var}(1.1Y) + 2\,\mathrm{Cov}(X, 1.1Y) = \mathrm{Var}(X) + 1.1^2\,\mathrm{Var}(Y) + 2 \cdot 1.1\,\mathrm{Cov}(X, Y).$$

   We are given that $\mathrm{Var}(X) = 5,000$, $\mathrm{Var}(Y) = 10,000$, so the only remaining unknown quantity is $\mathrm{Cov}(X, Y)$, which can be computed via the general formula for $\mathrm{Var}(X + Y)$:

   $$\mathrm{Cov}(X, Y) = \frac{1}{2}\left(\mathrm{Var}(X + Y) - \mathrm{Var}(X) - \mathrm{Var}(Y)\right) = \frac{1}{2}(17,000-5,000-10,000) = 1,000.$$

   Substituting this into the above formula, we get the answer:

   $$\mathrm{Var}(X + 1.1Y) = 5,000 + 1.1^2 \cdot 10,000 + 2 \cdot 1.1 \cdot 1,000 = 19,520$$

4. A company insures homes in three cities, J, K, L. The losses occurring in these cities are independent. The moment-generating functions for the loss distributions of the cities are

   $$M_J(t) = (1 - 2t)^{-3}, \quad M_K(t) = (1 - 2t)^{-2.5}, \quad M_L(t) = (1 - 2t)^{-4.5}$$

   Let X represent the combined losses from the three cities. Calculate $E(X^3)$.

   Solution: Let J, K, L denote the losses from the three cities. Then $X = J + K + L$.

Since J, K, L are independent, the moment-generating function for their sum, X, is equal to the product of the individual moment-generating functions, i.e.,

$$M_X(t) = M_K(t)M_J(t)M_L(t) = (1-2t)^{-3-2.5-4.5} = (1-2t)^{-10}.$$

Differentiating this function, we get

$$\begin{aligned}M'(t) &= (-2)(-10)(1-2t)^{-11},\\ M''(t) &= (-2)^2(-10)(-11)(1-2t)^{-12},\\ M'''(t) &= (-2)^3(-10)(-11)(-12)(1-2t)^{-13}.\end{aligned}$$

Hence, $E(X^3) = M_X'''(0) = (-2)^3(-10)(-11)(-12) = 10,560$.

5. Given that $E(X) = 5$, $E(X^2) = 27.4$, $E(Y) = 7$, $E(Y^2) = 51.4$ and $\operatorname{Var}(X+Y) = 8$, find $\operatorname{Cov}(X+Y, X+1.2Y)$.

Solution: By definition,

$$\operatorname{Cov}(X+Y, X+1.2Y) = E((X+Y)(X+1.2Y)) - E(X+Y)E(X+1.2Y).$$

Using the properties of expectation and the given data, we get

$$\begin{aligned}E(X+Y)E(X+1.2Y) &= (E(X)+E(Y))(E(X)+1.2E(Y)) = (5+7)(5+1.2\cdot 7) = 160.8,\\ E((X+Y)(X+1.2Y)) &= E(X^2) + 2.2E(XY) + 1.2E(Y^2)\\ &= 27.4 + 2.2E(XY) + 1.2\cdot 51.4 = 2.2E(XY) + 89.08,\\ \operatorname{Cov}(X+Y, X+1.2Y) &= 2.2E(XY) + 89.08 - 160.8 = 2.2E(XY) - 71.72\end{aligned}$$

To complete the calculation, it remains to find $E(XY)$. To this end we make use of the still unused relation $\operatorname{Var}(X+Y) = 8$:

$$\begin{aligned}8 = \operatorname{Var}(X+Y) &= E((X+Y)^2) - (E(X+Y))^2 = E(X^2) + 2E(XY) + E(Y^2) - (E(X)+E(Y))^2\\ &= 27.4 + 2E(XY) + 51.4 - (5+7)^2 = 2E(XY) - 65.2,\end{aligned}$$

so $E(XY) = 36.6$. Substituting this above gives $\operatorname{Cov}(X+Y, X+1.2Y) = 2.2\cdot 36.6 - 71.72 = 8.8$.