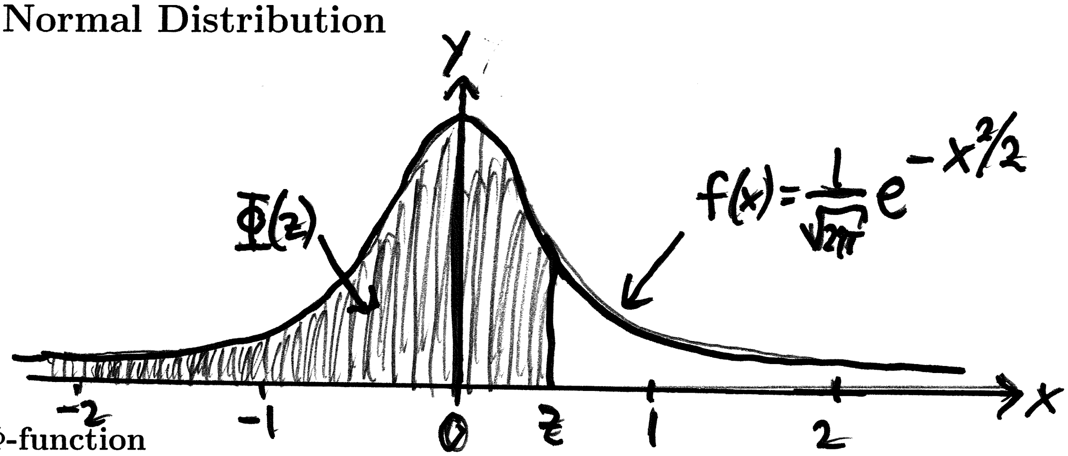
The Normal Distribution



The Φ-function

- **Definition:** $\Phi(z)$ is the area under the "Bell curve function" $(1/\sqrt{2\pi})e^{-x^2/2}$ between $-\infty$ to z, i.e., $\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-x^2/2} dt$ (see picture).

- **Probabilistic significance:** $\Phi(z)$ is the cumulative distribution function (c.d.f.) of the standard normal distribution (see below).

- **Properties:**

  (i) $\Phi(-\infty) = 0$, $\Phi(\infty) = 1$, $\Phi(z)$ is increasing;

  (ii) $\Phi(-x) = 1 - \Phi(x)$, $\Phi(0) = 0.5$.

  Properties (i) are just general properties of any c.d.f.; properties (ii) express the symmetry of Φ with respect to the y-axis.

- **Normal table:** Since there exists no "explicit" formula for $\Phi(x)$ (the integral representing $\Phi(x)$ cannot be evaluated in terms of elementary functions), for computations involving Φ one has to resort to tabulated values of $\Phi(x)$. Such a "normal table" can be found in the back of Hogg/Tanis, and will be provided in actuarial exams. Using calculators with built-in Φ function, or with integrating capabilities, is not allowed in actuarial exams.

- **Upper percentiles:** Upper percentiles (or "upper percent points") are defined just like ordinary percentiles, but with respect to the complementary probability. For example, the 5th upper percentile is the z-value for which $1 - \Phi(z)$ equals 0.05; it is denoted by $z_{0.05}$ and approximately equal to 1.64. The 5th upper percentile is the same as the 95th ordinary percentile defined as the z-value at which $\Phi(z) = 0.95$.

The standard normal distribution $N(0,1)$

- **Cumulative distribution function (c.d.f.):** $F(x) = \Phi(x)$, where Φ is the function defined above.
- **Density (p.d.f.):** $f(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}x^2\right) \quad (-\infty < x < \infty)$.
- **Expectation and variance:** $E(X) = 0$, $\mathrm{Var}(X) = 1$
- **Moment-generating function:** $M(t) = \exp\left(\frac{1}{2}t^2\right)$
- **Computation of probabilities:** If X is standard normal $N(0,1)$, then $P(a < X < b) = \Phi(b) - \Phi(a)$, $P(X < b) = \Phi(b)$, $P(X > a) = 1 - \Phi(a)$.

The general normal distribution $N(\mu, \sigma^2)$

- **Cumulative distribution function (c.d.f.):** $F(x) = \Phi\left(\frac{x-\mu}{\sigma}\right)$, where Φ is the function defined above.
- **Density (p.d.f.):** $f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right) \quad (-\infty < x < \infty)$.
- **Expectation and variance:** $E(X) = \mu$, $\mathrm{Var}(X) = \sigma^2$
- **Moment-generating function:** $M(t) = \exp\left(\mu t + \frac{1}{2}\sigma^2 t^2\right)$. (Note that the factor σ^2 here is in the numerator, not in the denominator, as in the formula for the p.d.f., $f(x)$.)
- **Standardizing a normal random variable:** Rescaling a normally distributed r.v. (with general values of μ and σ) by subtracting its mean and dividing by its standard deviation (i.e., σ, **not** σ^2) gives one that has standard normal distribution. In other words, if X is $N(\mu, \sigma^2)$, then $Z = (X - \mu)/\sigma$ is $N(0,1)$.
- **Computation of probabilities:** To compute probabilities involving a r.v. X with normal distribution $N(\mu, \sigma^2)$, first convert the probabilities into probabilities involving the standardized version of X, $Z = (X - \mu)/\sigma$, then express the latter probabilities via the Φ function as above. For example,

$$P(a < X < b) = P\left(\frac{a-\mu}{\sigma} < \frac{X-\mu}{\sigma} < \frac{b-\mu}{\sigma}\right) = P\left(\frac{a-\mu}{\sigma} < Z < \frac{b-\mu}{\sigma}\right)$$
$$= \Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right).$$