Moment-generating functions

Solutions

1. (May 2000 Exam, Problem 4-110 of Problemset 4) A company insures homes in three cities, J, K, L. The losses occurring in these cities are independent. The moment-generating functions for the loss distributions of the cities are

\[ M_J(t) = (1 - 2t)^{-3}, \quad M_K(t) = (1 - 2t)^{-2.5}, \quad M_L(t) = (1 - 2t)^{-4.5} \]

Let \( X \) represent the combined losses from the three cities. Calculate \( E(X^3) \).

**Solution:** Let \( J, K, L \) denote the losses from the three cities. Then \( X = J + K + L \). Since \( J, K, L \) are independent, the moment-generating function for their sum, \( X \), is equal to the product of the individual moment-generating functions, i.e.,

\[ M_X(t) = M_K(t)M_J(t)M_L(t) = (1 - 2t)^{-3 - 2.5 - 4.5} = (1 - 2t)^{-10}. \]

Differentiating this function, we get

\[ M'_X(t) = (-2)(-10)(1 - 2t)^{-11}, \quad M''_X(t) = (-2)^2(-10)(-11)(1 - 2t)^{-12}, \quad M'''_X(t) = (-2)^3(-10)(-11)(-12)(1 - 2t)^{-13}. \]

Hence, \( E(X^3) = M'''_X(0) = (-2)^3(-10)(-11)(-12) = 10,560 \).

2. (August 2007 Exam) A loss for a company has moment-generating function \( M(t) = 0.16/(0.16 - t), \) \( t < 0.16 \). An insurance policy pays a benefit equal to 70% of the loss. What is the moment-generating function of the benefit?

**Solution:** If \( X \) denotes the loss, and \( Y \) the benefit, then \( Y = 0.7X \). Thus,

\[ M_Y(t) = E(e^{tY}) = E(e^{t0.7X}) = M_X(0.7t) = \frac{0.16}{0.16 - 0.7t}. \]

3. (Cf. Problem 2.5.21 in Hogg/Tanis) Given that \( X \) has moment-generating function

\[ M(t) = \frac{1}{6}e^{-2t} + \frac{1}{3}e^{-t} + \frac{1}{4}e^t + \frac{1}{4}e^{2t}, \]

find \( P(|X| \leq 1) \).

**Solution:** Comparing the given formula for \( M(t) \) with the general formula for a the mgf of a discrete distribution, \( M_X(t) = \sum_x f(x)e^{tx} \), where the sum is over all values \( x \) of \( X \), we see that \( X \) must have values \(-2, -1, 1, \) and \( 2 \), with probabilities \( 1/6, 1/3, 1/4, \) and \( 1/4 \), respectively. Thus, \( P(|X| \leq 1) = P(X = -1) + P(X = 1) = 1/3 + 1/4 = 7/12 \).
4. (May 2007 Exam) Suppose that \( M(t) \) is a moment-generating function of some random variable. Which of the following are moment-generating functions of some (other) random variables?
(i) \( M(t)M(5t) \); (ii) \( 2M(t) \); (iii) \( e^{-t}M(t) \).

**Solution:** The second function, \( 2M(t) \), can be eliminated since at \( t = 0 \) it equals \( 2M(0) = 2 \cdot 1 = 2 \), whereas any mgf must be equal to 1 at \( t = 0 \).

The other two functions, however, are mgfs of suitable random variables. This hinges on the following facts:

1. The product of any two mgf’s is again an mgf (for some r.v.).
2. If \( M(t) \) is an mgf, then so is \( M(ct) \) for any non-zero constant \( c \).
3. The function \( e^{-t} \) is an mgf.

To see (1), suppose \( M_1(t) \) and \( M_2(t) \) are the mgf’s of random variable \( X_1 \) and \( X_2 \). One can, without loss of generality, assume that \( X_1 \) and \( X_2 \) are independent. Then \( X_1 + X_2 \) has mgf \( M_1(t)M_2(t) \).

To see (2), let \( M(t) \) be the mgf of \( X \), and let \( Y = cX \). Then \( M_Y(t) = E(e^{tY}) = E(e^{tcX}) = M(tc) = M(ct) \), so the function \( M(ct) \) is the mgf of \( Y \).

To see (3), simply take \( X \) to be the random variable that takes on the single value \(-1\) with probability 1. Then \( X \) has mgf \( 1 \cdot e^{-t} \).