Moment-generating functions

Solutions

1. (May 2000 Exam, Problem 4-110 of Problemset 4) A company insures homes in three cities, J, K, L. The losses occurring in these cities are independent. The moment-generating functions for the loss distributions of the cities are

$$M_J(t) = (1-2t)^{-3}, \quad M_K(t) = (1-2t)^{-2.5}, \quad M_L(t) = (1-2t)^{-4.5}$$

Let X represent the combined losses from the three cities. Calculate $E(X^3)$.

Solution: Let J, K, L denote the losses from the three cities. Then $X = J + K + L$. Since J, K, L are independent, the moment-generating function for their sum, X, is equal to the product of the individual moment-generating functions, i.e.,

$$M_X(t) = M_K(t)M_J(t)M_L(t) = (1-2t)^{-3-2.5-4.5} = (1-2t)^{-10}.$$

Differentiating this function, we get

$$\begin{aligned} M'(t) &= (-2)(-10)(1-2t)^{-11}, \\ M''(t) &= (-2)^2(-10)(-11)(1-2t)^{-12}, \\ M'''(t) &= (-2)^3(-10)(-11)(-12)(1-2t)^{-13}. \end{aligned}$$

Hence, $E(X^3) = M_X'''(0) = (-2)^3(-10)(-11)(-12) = 10,560$.

2. (August 2007 Exam) A loss for a company has moment-generating function $M(t) = 0.16/(0.16-t)$, $t < 0.16$. An insurance policy pays a benefit equal to 70% of the loss. What is the moment-generating function of the benefit?

Solution: If X denotes the loss, and Y the benefit, then $Y = 0.7X$. Thus,

$$M_Y(t) = E(e^{tY}) = E(e^{t\cdot 0.7Y}) = M_X(0.7t) = \frac{0.16}{0.16-0.7t}.$$

3. (Cf. Problem 2.5.21 in Hogg/Tanis) Given that X has moment-generating function

$$M(t) = \frac{1}{6}e^{-2t} + \frac{1}{3}e^{-t} + \frac{1}{4}e^{t} + \frac{1}{4}e^{2t},$$

find $P(|X| \le 1)$.

Solution: Comparing the given formula for $M(t)$ with the general formula for a the mgf of a discrete distribution, $M_X(t) = \sum_x f(x)e^{tx}$, where the sum is over all values x of X, we see that X must have values -2, -1, 1, and 2, with probabilities 1/6, 1/3, 1/4, and 1/4, respectively. Thus, $P(|X| \le 1) = P(X = -1) + P(X = 1) = 1/3 + 1/4 = 7/12$.

4. (May 2007 Exam) Suppose that $M(t)$ is a moment-generating function of some random variable. Which of the following are moment-generating functions of some (other) random variables?
(i) $M(t)M(5t)$; (ii) $2M(t)$; (iii) $e^{-t}M(t)$.

Solution: The second function, $2M(t)$, can be eliminated since at $t = 0$ it equals $2M(0) = 2 \cdot 1 = 2$, whereas any mgf must be equal to 1 at $t = 0$.

The other two functions, however, are mgfs of suitable random variables. This hinges on the following facts:

(1) The product of any two mgf's is again an mgf (for some r.v.).

(2) If $M(t)$ is an mgf, then so is $M(ct)$ for any non-zero constant c.

(3) The function e^{-t} is an mgf.

To see (1), suppose $M_1(t)$ and $M_2(t)$ are the mgf's of random variable X_1 and X_2. One can, without loss of generality, assume that X_1 and X_2 are independent. Then $X_1 + X_2$ has mgf $M_1(t)M_2(t)$.

To see (2), let $M(t)$ be the mgf of X, and let $Y = cX$. Then $M_Y(t) = E(e^{tY}) = E(e^{tcX}) = M(tc) = M(ct)$, so the function $M(ct)$ is the mgf of Y.

To see (3), simply take X to be the random variable that takes on the single value -1 with probability 1. Then X has mgf $1 \cdot e^{-t}$.