Moment-generating functions

Definitions and properties

- **General definition of an mgf:** $M(t) = M_X(t) = \mathrm{E}(e^{tX})$.
- **Mgf of discrete r.v.'s:** $M(t) = \sum e^{tx} f(x)$, where $f(x)$ is the p.m.f. of X and the sum is taken over all values x of X.
- **Mgf of continuous r.v.'s:** $M(t) = \int_*^* e^{tx} f(x) dx$, where $f(x)$ is the p.d.f. of X, and the range of integration is the range on which $f(x)$ is defined.
- **Expectation, variance, and moments via mgf's:** $M(0) = 1$, $M'(0) = \mathrm{E}(X)$, $M''(0) = \mathrm{E}(X^2)$, $M'''(0) = \mathrm{E}(X^3)$, etc.; $\mathrm{Var}(X) = M''(0) - M'(0)^2$.
- **Mgf of a multiple of a r.v.:** If X has mgf $M_X(t)$, and $Y = cX$ with c a constant, then the mgf of Y is $M_Y(t) = E(e^{tY}) = E(e^{tcX}) = M_X(tc)$.
- **Mgf of a sum of independent r.v.'s X and Y:** If X and Y are independent, then $X + Y$ has mgf $M_{X+Y}(t) = M_X(t)M_Y(t)$. (An analogous formula holds for sums of more than two independent r.v.'s.)

Notes

- Note that the product formula for mgf's involves the *sum* of independent r.v.'s, not the product. The reason behind this is that the definition of the mgf of $X+Y$ is the expectation of $e^{t(X+Y)}$, which is equal to the product $e^{tX} \cdot e^{tY}$. In case of independence, the expectation of that product is the product of the expectations.
- Note that in order to apply mgf's to compute expectations, variances, etc., an mgf only has to be defined for *small* values t, since the critical range is that when t is close to 0. In fact, for some continuous distributions (e.g., exponential) the integral defining an mgf only converges for values t below certain (positive) bound.

Practice Problems

1. (May 2000 Exam, Problem 4-110 of Problemset 4) A company insures homes in three cities, J, K, L. The losses occurring in these cities are independent. The moment-generating functions for the loss distributions of the cities are

$$M_J(t) = (1-2t)^{-3}, \quad M_K(t) = (1-2t)^{-2.5}, \quad M_L(t) = (1-2t)^{-4.5}$$

   Let X represent the combined losses from the three cities. Calculate $E(X^3)$.

2. (August 2007 Exam) A loss for a company has moment-generating function $M(t) = 0.16/(0.16-t)$, $t < 0.16$. An insurance policy pays a benefit equal to 70% of the loss. What is the moment-generating function of the benefit?

3. (Cf. Problem 2.5.21 in Hogg/Tanis) Given that X has moment-generating function

$$M(t) = \frac{1}{6}e^{-2t} + \frac{1}{3}e^{-t} + \frac{1}{4}e^{t} + \frac{1}{4}e^{2t},$$

   find $P(|X| \le 1)$.

4. (May 2007 Exam) Suppose that $M(t)$ is a moment-generating function of some random variable. Which of the following are moment-generating functions of some (other) random variables?
   (i) $M(t)M(5t)$; (ii) $2M(t)$; (iii) $e^{-t}M(t)$.