Tips for computing integrals

Computing probabilities, expectations, etc., for continuous random variables usually boils down to evaluating integrals. Here are some some tips for evaluating integrals that typically arise in this context.

- **Integrals over functions defined via case distinctions:** In probability applications it is very common to encounter integrals in which the integrand is given by different formulas for different ranges of x-values. You then have to break up the integral into separate integrals over these ranges.
- **Integrals over functions involving** $\max(\dots)$ **or** $\min(\dots)$**:** This can be handled by a case distinction. For example, if a quantity is defined as $\min(1, x/2)$ (a typical situation when "caps" are involved), then it is equal to 1 if $x > 2$, and equal to $x/2$ if $x \le 2$.
- **Integrals over functions involving absolute values:** The absolute value function cannot be directly integrated. You need to break up the range of integration according to the sign of the expression inside the absolute value, and replace the absolute value by plus or minus the expression inside, depending on the range you are in. For example, if the integrand involves $|x-2|$, then you need to consider separately the range $x > 2$ (where $|x-2|$ can be replaced by $x-2$) and $x \le 2$ (where $|x-2|$ can be replaced by $-(x-2)$).
- **Watch out for sign mistakes:** It is easy to make a sign mistake, especially in integrals involving exponential densities. For example, the integral $\int_2^\infty e^{-2x} dx$ is equal to $(-1/2)e^{-2x}\Big|_2^\infty$, which is $(-1/2)(0 - e^{-4}) = (1/2)e^{-4}$, not $(-1/2)e^{-4}$. To guard against such mistakes, always do a "sanity check": ask yourself if the result makes sense in the context of the given problem. For example, if you get a negative value for an integral over a positive function, it obviously cannot be right.
- **Some important integration techniques:**
  - **Integration by parts:** Typical examples in actuarial problems are exponential integrals such as $\int xe^{-x} dx$.
  - **Integration by substitution:** Typical examples are $\int xe^{-x^2} dx$ (substitute $u = x^2$), or $\int x\sqrt{x^2+1}$ (substitute $u = x^2 + 1$). When using substitution, make sure to also substitute the limits.
  - **Trig integrals:** Integrals over trig and inverse trig functions are among the most difficult "doable" integrals. Fortunately, those integrals virtually never arise in in probabilistic applications, so you need not know the various tricks associated with such integrals.
- **Some useful integrals:** The following are some formulas for integrals that are worth memorizing. The integrals are not difficult to evaluate directly, but they arise frequently, and having these memorized will save you time.
  - $\int_0^1 x^c dx = \frac{1}{c+1}$ $(c > -1)$
  - $\int_1^\infty \frac{1}{x^c} dx = \frac{1}{c-1}$ $(c > 1)$
  - $\int_0^\infty e^{-cx} dx = \frac{1}{c}$ $(c > 0)$
  - $\int_a^\infty e^{-cx} dx = \frac{e^{-ca}}{c}$ $(c > 0,\ a > 0)$