Problems on general probability rules, independence, conditional probability

1. Assuming $A$, $B$, $C$ are mutually independent, with $P(A) = P(B) = P(C) = 0.1$, compute:

   (a) $P(A \cup B)$  \textbf{Solution:} $P(A) + P(B) - P(A)P(B) = 0.19$
   
   (b) $P(A \cup B \cup C)$  
   \textbf{Solution:} By formula the formula for $P(A \cup B \cup C)$ and indep., $P(A \cup B \cup C) = 3 \cdot 0.1 - 3 \cdot 0.1^2 + 0.1^3 = 0.271$
   
   (c) $P(A \setminus (B \cup C))$  
   \textbf{Solution:} $P(A) - P(A \cap B) - P(A \cap C) + P(A \cap B \cap C) = 0.081$

2. Given that $P(A) = 0.3$, $P(A|B) = 0.4$, and $P(B) = 0.5$, compute:

   (a) $P(A \cap B)$  \textbf{Solution:} $P(A|B)P(B) = 0.4 \cdot 0.5 = 0.2$
   
   (b) $P(B|A)$  \textbf{Solution:} $P(B \cap A)/P(A) = 0.2/0.3 = 0.666$
   
   (c) $P(A'|B)$  \textbf{Solution:} $P(A' \cap B)/P(B) = ((P(B) - P(A \cap B))/P(B) = 0.6$
   
   (d) $P(A|B')$  \textbf{Solution:} $P(A \cap B')/P(B') = (P(A) - P(A \cap B))/(1 - P(B)) = 0.2$

3. Assume $A$ and $B$ are independent events with $P(A) = 0.2$ and $P(B) = 0.3$. Let $C$ be the event that \textbf{at least one} of $A$ or $B$ occurs, and let $D$ be the event that \textbf{exactly one} of $A$ or $B$ occurs.

   (a) Find $P(C)$.
   \textbf{Solution:} The event $C$ is just the union of $A$ and $B$, so $P(C) = P(A \cup B) = P(A) + P(B) - P(A)P(B) = 0.44$
(b) Find \( P(D) \).

**Solution:** Drawing a Venn diagram, we see that \( D \) consists of the union of \( A \) and \( B \) minus the overlap. Thus, \( P(D) = P(A \cup B \setminus A \cap B) = P(A \cup B) - P(A)P(B) = 0.38 \)

(c) Find \( P(A|D) \) and \( P(D|A) \).

**Solution:** \( P(A|D) = P(A \cap D)/P(D) = P(A \setminus A \cap B)/P(D) = (0.2 - 0.2 \cdot 0.3)/0.38 = \frac{7}{19} \) \( P(D|A) = P(A \setminus A \cap B)/P(A) = (0.2 - 0.2 \cdot 0.5)/0.2 = 0.7 \)

(d) Determine whether \( A \) and \( D \) are independent.

**Solution:** \( A \) and \( D \) are not independent since by the previous part, \( P(A|D) \neq P(A) \).

**Alternative solution:** From above, \( P(A \cap D) = 0.14, P(A)P(D) = 0.2 \cdot 0.38 = 0.076 \), so \( P(A \cap D) \neq P(A)P(D) \), and therefore \( A \) and \( D \) are not independent.

4. Given that \( P(A \cup B) = 0.7 \) and \( P(A \cup B') = 0.9 \), find \( P(A) \).

**Solution:** By De Morgan’s law, \( P(A' \cap B') = P((A \cup B)') = 1 - P(A \cup B) = 1 - 0.7 = 0.3 \) and similarly \( P(A' \cap B) = 1 - P(A \cup B') = 1 - 0.9 = 0.1 \). Thus, \( P(A') = P(A' \cap B') + P(A' \cap B) = 0.3 + 0.1 = 0.4 \), so \( P(A) = 1 - 0.4 = 0.6 \).

5. Given that \( A \) and \( B \) are independent with \( P(A) = 2P(B) \) and \( P(A \cap B) = 0.15 \), find \( P(A' \cap B') \).

**Solution:** By independence and the given data, \( 0.15 = P(A \cap B) = P(A)P(B) = 2P(B)^2 \), so \( P(B) = \sqrt{0.075} = 0.273 \), and \( P(A) = 2P(B) = 0.546 \). Hence \( P(A' \cap B') = P(A')P(B') = (1 - 0.546)(1 - 0.273) = 0.33 \). (Note the use of the “independence of complements” property here.)

6. Given that \( A \) and \( B \) are independent with \( P(A \cup B) = 0.8 \) and \( P(B') = 0.3 \), find \( P(A) \).

**Solution:** We have \( P(B) = 1 - 0.3 = 0.7 \) and \( 0.8 = P(A) + P(B) - P(A \cap B) = P(A) + P(B) - P(A)P(B) = P(A)(1 - 0.7) + 0.7 \). Solving for \( P(A) \) gives \( P(A) = (0.8 - 0.7)/0.3 = 0.33 \).
7. Given that \( P(A) = 0.2, P(B) = 0.7, \) and \( P(A|B) = 0.15, \) find \( P(A' \cap B'). \)

**Solution:** By De Morgan’s Law, 
\[ P(A' \cap B') = P((A \cup B)') = 1 - P(A \cup B) = 1 - P(A) - P(B) + P(A \cap B). \]
Using the given values of \( P(A) \) and \( P(B) \) and \( P(A \cap B) = P(A|B)P(B) = 0.15 \cdot 0.7 = 0.105 \) (the multiplication formula), we get 
\[ P(A' \cap B') = 1 - 0.2 - 0.7 + 0.105 = 0.205. \]

8. Given \( P(A) = 0.6, P(B) = 0.7, P(C) = 0.8, P(A \cap B) = 0.3, P(A \cap C) = 0.4, P(B \cap C) = 0.5, P(A \cap B \cap C) = 0.2, \) find \( P(A' \cap B' \cap C'). \)

**Solution:** If \( A', B' \) and \( C' \) were independent, we could apply the product formula, and the answer would be immediate, but we don’t know this (in fact, they are not). However, from a Venn diagram we see that 
\[ P(A \cap B' \cap C') = P(A) - P(A \cap B) - P(A \cap C) + P(A \cap B \cap C). \]
Inserting the given values, we get 
\[ 0.6 - 0.3 - 0.4 + 0.2 = 0.1 \] as answer.