Problems on general probability rules, independence, conditional probability

1. Assuming A, B, C are mutually independent, with $P(A) = P(B) = P(C) = 0.1$, compute:

   (a) $P(A \cup B)$ **Solution:** $P(A) + P(B) - P(A)P(B) = \boxed{0.19}$

   (b) $P(A \cup B \cup C)$
   Solution: By formula the formula for $P(A \cup B \cup C)$ and indep., $P(A \cup B \cup C) = 3 \cdot 0.1 - 3 \cdot 0.1^2 + 0.1^3 = \boxed{0.271}$

   (c) $P(A \setminus (B \cup C))$
   Solution: $P(A) - P(A \cap B) - P(A \cap C) + P(A \cap B \cap C) = \boxed{0.081}$

2. Given that $P(A) = 0.3$, $P(A|B) = 0.4$, and $P(B) = 0.5$, compute:

   (a) $P(A \cap B)$ **Solution:** $P(A|B)P(B) = 0.4 \cdot 0.5 = \boxed{0.2}$

   (b) $P(B|A)$ **Solution:** $P(B \cap A)/P(A) = 0.2/0.3 = \boxed{0.666}$

   (c) $P(A'|B)$ **Solution:** $P(A' \cap B)/P(B) = ((P(B) - P(A \cap B))/P(B) = \boxed{0.6}$

   (d) $P(A|B')$ **Solution:** $P(A \cap B')/P(B') = (P(A) - P(A \cap B))/(1 - P(B)) = \boxed{0.2}$

3. Assume A and B are independent events with $P(A) = 0.2$ and $P(B) = 0.3$. Let C be the event that **at least one** of A or B occurs, and let D be the event that **exactly one** of A or B occurs.

   (a) Find $P(C)$.
   Solution: The event C is just the union of A and B, so $P(C) = P(A \cup B) = P(A) + P(B) - P(A)P(B) = \boxed{0.44}$

(b) Find $P(D)$.

Solution: Drawing a Venn diagram, we see that D consists of the union of A and B minus the overlap. Thus, $P(D) = P(A \cup B \setminus A \cap B) = P(A \cup B) - P(A)P(B) = \boxed{0.38}$

(c) Find $P(A|D)$ and $P(D|A)$.

Solution: $P(A|D) = P(A \cap D)/P(D) = P(A \setminus A \cap B)/P(D) = (0.2 - 0.2 \cdot 0.3)/0.38 = \boxed{7/19}$. $P(D|A) = P(A \setminus A \cap B)/P(A) = (0.2 - 0.2 \cdot 0.5)/0.2 = \boxed{0.7}$.

(d) Determine whether A and D are independent.

Solution: A and D are not independent since by the previous part, $P(A|D) \neq P(A)$.

Alternative solution: From above, $P(A \cap D) = 0.14$, $P(A)P(D) = 0.2 \cdot 0.38 = 0.076$, so $P(A \cap D) \neq P(A)P(D)$, and therefore A and D are not independent.

4. Given that $P(A \cup B) = 0.7$ and $P(A \cup B') = 0.9$, find $P(A)$.

Solution: By De Morgan's law, $P(A' \cap B') = P((A \cup B)') = 1 - P(A \cup B) = 1 - 0.7 = 0.3$ and similarly $P(A' \cap B) = 1 - P(A \cup B') = 1 - 0.9 = 0.1$. Thus, $P(A') = P(A' \cap B') + P(A' \cap B) = 0.3 + 0.1 = 0.4$, so $P(A) = 1 - 0.4 = \boxed{0.6}$.

5. Given that A and B are independent with $P(A) = 2P(B)$ and $P(A \cap B) = 0.15$, find $P(A' \cap B')$.

Solution: By independence and the given data, $0.15 = P(A \cap B) = P(A)P(B) = 2P(B)^2$, so $P(B) = \sqrt{0.075} = 0.273$, and $P(A) = 2P(B) = 0.546$. Hence $P(A' \cap B') = P(A')P(B') = (1 - 0.546)(1 - 0.273) = \boxed{0.33}$. (Note the use of the "independence of complements" property here.)

6. Given that A and B are independent with $P(A \cup B) = 0.8$ and $P(B') = 0.3$, find $P(A)$.

Solution: We have $P(B) = 1 - 0.3 = 0.7$ and $0.8 = P(A) + P(B) - P(A \cap B) = P(A) + P(B) - P(A)P(B) = P(A)(1 - 0.7) + 0.7$. Solving for $P(A)$ gives $P(A) = (0.8 - 0.7)/0.3 = \boxed{0.33}$.

7. Given that $P(A) = 0.2$, $P(B) = 0.7$, and $P(A|B) = 0.15$, find $P(A' \cap B')$.

   Solution: By De Morgan's Law, $P(A' \cap B') = P((A \cup B)') = 1 - P(A \cup B) = 1 - P(A) - P(B) + P(A \cap B)$. Using the given values of $P(A)$ and $P(B)$ and $P(A \cap B) = P(A|B)P(B) = 0.15 \cdot 0.7 = 0.105$ (the multiplication formula), we get $P(A' \cap B') = 1 - 0.2 - 0.7 + 0.105 = \boxed{0.205}$.

8. Given $P(A) = 0.6$, $P(B) = 0.7$, $P(C) = 0.8$, $P(A \cap B) = 0.3$, $P(A \cap C) = 0.4$, $P(B \cap C) = 0.5$, $P(A \cap B \cap C) = 0.2$, find $P(A \cap B' \cap C')$.

   Solution: If A, B' and C' were independent, we could apply the product formula, and the answer would be immediate, but we don't know this (in fact, they are not). However, from a Venn diagram we see that $P(A \cap B' \cap C')$ is equal to to $P(A) - P(A \cap B) - P(A \cap C) + P(A \cap B \cap C)$. Inserting the given values, we get $0.6 - 0.3 - 0.4 + 0.2 = \boxed{0.1}$ as answer.