Practice Problems: Proofs and Counterexamples involving Functions

The following problems serve two goals: (1) to practice proof writing skills in the context of abstract function properties; and (2) to develop an intuition, and “feel” for properties like injective, increasing, bounded, etc., so that you can easily come up with a “guess” whether a statement is likely true, and find counterexamples for false statements. None of the problems is particularly difficult: The proofs for true statements are all quite routine, and counterexamples for false statements are not hard to discover once you have a good intuitive understanding of the definitions. Try to master them all!

1. **Proofs involving surjective and injective properties of general functions**: Let $f : A \to B$ and $g : B \to C$ be functions, and let $h = g \circ f$ be the composition of $g$ and $f$. For each of the following statements, either give a formal proof or counterexample. (A counterexample means a specific example of sets $A, B, C$ and functions $f : A \to B$, and $g : B \to C$, for which the statement is false.)

   (a) If $f$ and $g$ are injective, then $h$ is injective.
   (b) If $f$ and $g$ are surjective, then $h$ is surjective.
   (c) If $h$ is injective, then $f$ is injective.
   (d) If $h$ is injective, then $g$ is injective.
   (e) If $h$ is surjective, then $f$ is surjective.
   (f) If $h$ is surjective, then $g$ is surjective.

**Sol:**

(a): If $f$ and $g$ are injective, then $h$ is injective.

*Proof:*

Suppose $f$ and $g$ are injective.

We seek to show that $h = g \circ f$ is injective.

Let $x, y \in A$ be given and assume $h(x) = h(y)$.

Since $h = g \circ f$, the last equation can written as $g(f(x)) = g(f(y))$.

Since $g$ is injective, this implies $f(x) = f(y)$.

Since $f$ is injective, it follows that $x = y$.

Summarizing, we have shown that, for any $x, y \in A$, $h(x) = h(y)$ implies $x = y$.

Therefore, $h$ is injective.

(b): If $f$ and $g$ are surjective, then $h$ is surjective.

*Proof:*

Suppose $f$ and $g$ are surjective.

We seek to show that $h = g \circ f$ is surjective.

Let $c \in C$ be given.

Since $g : B \to C$ is surjective and $c \in C$, there exists $b \in B$ such that $g(b) = c$.

Since $f : A \to B$ is surjective and $b \in B$, there exists $a \in A$ such that $f(a) = b$.

Combining these equations, we get $c = g(b) = g(f(a)) = h(a)$.

Summarizing, we have shown that, for any $c \in C$, there exists $a \in A$ such that $h(a) = c$.

Therefore, $h$ is surjective.

(c): If $h$ is injective, then $f$ is injective. **TRUE.**

*Proof:*

...
Suppose \( h \) is injective.
We seek to show that \( f \) is injective.
Let \( x, y \in A \) be given and assume \( f(x) = f(y) \).
Then \( g(f(x)) = g(f(y)) \).
Since \( h = g \circ f \), this can be written as \( h(x) = h(y) \).
Since \( h \) is injective, the latter equation implies \( x = y \).
Summarizing, we have shown that for all \( x, y \in A \), \( f(x) = f(y) \) implies \( x = y \).
Therefore \( f \) is injective.

(d): If \( h \) is injective, then \( g \) is injective. FALSE.
Counterexample: Let \( A = \{1\} \), \( B = \mathbb{R} \), \( C = \{1\} \), and define \( f : \{1\} \to \mathbb{R} \) by \( f(1) = 1 \), and \( g : \mathbb{R} \to \{1\} \) by \( g(x) = 1 \) for all \( x \in \mathbb{R} \). With this choice, \( h \) is a function from \( \{1\} \) to \( \{1\} \), defined by \( h(1) = g(f(1)) = g(1) = 1 \). The function \( h \) is injective (since it maps the single element 1 in \( A \) to the single element 1 in \( C \)), but \( g \) is not (since it maps every real number to 1).

(e): If \( h \) is surjective, then \( f \) is surjective. FALSE.
Counterexample: The counterexample constructed in the previous part is also a counterexample here: The function \( h : \{1\} \to \{1\} \) is a bijection and thus surjective, but \( f : \{1\} \to \mathbb{R} \) is not surjective (its only value in \( \mathbb{R} \) is the number 1).

(f): If \( h \) is surjective, then \( g \) is surjective. TRUE.
Proof:
Suppose \( h \) is surjective.
We seek to show that \( g \) is surjective.
Let \( z \in C \) be given.
Since \( h \) is surjective, there exists an \( x \in A \) such that \( h(x) = z \).
Let \( y = f(x) \).
Then \( g(y) = g(f(x)) = h(x) = z \).
Also, since \( f \) is a function from \( A \) to \( B \), we have \( y = f(x) \in B \).
Summarizing, we have shown that, for any element \( z \in C \) there exists an element \( y \in B \) such that \( g(y) = z \).
Therefore \( g \) is surjective.

2. Proofs involving bounded functions: Let \( f \) and \( g \) be functions from \( \mathbb{R} \) to \( \mathbb{R} \). For each of the following statements, either give a formal proof or a counterexample. (Here \( f + g \) is the function defined by \((f + g)(x) = f(x) + g(x)\) for all \( x \in \mathbb{R} \), \( fg \) is defined analogously.)

(a) If \( f \) and \( g \) are bounded, then so is \( f + g \).
(b) If \( f \) and \( g \) are bounded, then so is \( fg \).
(c) If \( f + g \) is bounded, then so are \( f \) and \( g \).
(d) If \( fg \) is bounded, then so are \( f \) and \( g \).
(e) If \( f \) is bounded, then so is the function \( e^f \).

**Sol:** (a), (b), and (e) are true, while (c) and (d) are false. We will give a proof for (a), and counterexamples for (c) and (d). The other assertions can be proved similarly.

(a): If \( f \) and \( g \) are bounded, then so is \( f + g \). **TRUE.**

**Proof:** Suppose \( f \) and \( g \) are bounded. By the definition of a bounded function, this means that there are constants \( M_1, M_2 \in \mathbb{R} \) such that

\[
(1) \quad |f(x)| \leq M_1 \quad \text{for all } x \in \mathbb{R},
\]

\[
(2) \quad |g(x)| \leq M_2 \quad \text{for all } x \in \mathbb{R}.
\]

It follows that, for all \( x \in \mathbb{R} \),

\[
|(f + g)(x)| = |f(x) + g(x)| \quad \text{(by definition of } f + g) \leq |f(x)| + |g(x)| \quad \text{(by triangle inequality } |a + b| \leq |a| + |b|) \leq M_1 + M_2 \quad \text{(by (1) and (2)).}
\]

Thus, \( |(f + g)(x)| \leq M_1 + M_2 \) for all \( x \in \mathbb{R} \). Therefore, \( f + g \) is bounded, with \( M = M_1 + M_2 \) as a bound.

(c): If \( f + g \) is bounded, then so are \( f \) and \( g \). **FALSE.**

**Counterexample:** Let \( f \) and \( g \) be defined by \( f(x) = x \) and \( g(x) = -x \) for all \( x \). Then \( (f + g)(x) = f(x) + g(x) = x + (-x) = 0 \) for all \( x \), so \( f + g \) is bounded, while neither \( f \) nor \( g \) are bounded.

(d): If \( fg \) is bounded, then so are \( f \) and \( g \). **FALSE.**

**Counterexample:** Let \( f \) and \( g \) be defined by \( f(x) = x \) and \( g(x) = 0 \) for all \( x \in \mathbb{R} \). Then \( (fg)(x) = f(x)g(x) = x \cdot 0 = 0 \) for all \( x \), so \( fg \) is bounded, but \( f \) is not bounded.

(e): If \( f \) is bounded, then so is the function \( e^f \). **TRUE.**

**Proof:** Suppose \( f \) is bounded. Then there exists \( M \in \mathbb{R} \) such that \( |f(x)| \leq M \) for all \( x \in \mathbb{R} \). Therefore \( -M \leq f(x) \leq M \) for all \( x \in \mathbb{R} \). By the properties of the exponential function it follows that \( e^{-M} \leq e^{f(x)} \leq e^M \) for all \( x \in \mathbb{R} \). Hence \( 0 < e^{f(x)} \leq e^M \) for all \( x \in \mathbb{R} \). Thus the function \( e^{f(x)} \) is bounded with bound \( M_1 = e^M \).

3. **Relations between various properties:** Let \( f \) be a function from \( \mathbb{R} \) to \( \mathbb{R} \). For each of the following statements, either give a formal proof or a counterexample.

(a) If \( f \) is surjective, then \( f \) is unbounded.
(b) If \( f \) is unbounded, then \( f \) is surjective.
(c) If \( f \) is increasing, then \( f \) is injective.
(d) If \( f \) is increasing, then \( f \) has an inverse.

**Sol:** (a): If \( f \) is surjective, then \( f \) is unbounded. **TRUE.**

**Proof:**
We prove the given statement by proving the contrapositive of this statement, namely:

(∗) If \( f \) is a bounded function from \( \mathbb{R} \) to \( \mathbb{R} \), then \( f \) is not surjective.

Assume \( f \) is a bounded function from \( \mathbb{R} \) to \( \mathbb{R} \).
By the definition of a bounded function, there exists \( M \in \mathbb{R} \) such that \( |f(x)| \leq M \) for all \( x \in \mathbb{R} \).
Let \( y = M + 1 \).
Then, for all \( x \in \mathbb{R} \), \( |f(x)| \leq M < M + 1 = y \), so \( |f(x)| < y \) and therefore \( f(x) \neq y \).
Thus, \( f(x) \) does not take on the value \( y \), and hence is not surjective.
This proves the statement (∗).
By contraposition, this is equivalent to the given statement.

(b): If \( f \) is unbounded, then \( f \) is surjective. FALSE.

Counterexample: \( f(x) = x^2 \). This function is unbounded, but not surjective since it does not take on negative values.

(c): If \( f \) is increasing, then \( f \) is injective. TRUE.

Proof.

Suppose \( f : \mathbb{R} \to \mathbb{R} \) is increasing.
Let \( x, y \in \mathbb{R} \) be given such that \( x \neq y \).
Then \( x < y \) or \( x > y \).
If \( x < y \), then \( f(x) < f(y) \) by the definition of an increasing function. Similarly, if \( x > y \), then \( f(y) > f(x) \).
Thus, in either case, we have \( f(x) \neq f(y) \).
Hence \( x \neq y \) implies \( f(x) \neq f(y) \).
Therefore \( f \) is injective.

(d): If \( f \) is increasing, then \( f \) has an inverse. FALSE.

Counterexample: \( f(x) = e^x \). This is a function from \( \mathbb{R} \) to \( \mathbb{R} \). It is increasing, but not surjective since \( f(x) = e^x > 0 \) for all \( x \), and so \( f(x) \) does not take on negative values. Hence \( f \) is not a bijection and thus does not have an inverse.