1. For the following questions an answer is sufficient: Just state the requested definition, theorem, formula, etc. Be sure to use correct notation, include any necessary quantifiers in the appropriate order, and connecting words (e.g., ”such that”) if necessary.

(a) The graph of a function \( f : A \rightarrow B \) is defined as the set \( \{(a,b) \in A \times B : b = f(a)\} \) or, equivalently, \( \{(a,f(a)) : a \in A\} \).

Solution: \( \{(a,b) \in A \times B : b = f(a)\} \) or, equivalently, \( \{(a,f(a)) : a \in A\} \).

(b) Evaluate the series \( \sum_{k=0}^{\infty} x^{3k+1} \), where \(|x| < 1\).

Solution: \( x \sum_{k=0}^{\infty} (x^3)^k = \frac{x}{1-x^3} \).

(c) Evaluate \( \prod_{i=1}^{n} n^i \).

Solution: \( \prod_{i=1}^{n} n^i = n \sum_{i=1}^{n} i = n^{n+1}/2. \)

(d) Let \( S \) be a set, and let \( A_1, A_2, \ldots, A_n \) be a partition of \( S \) into \( n \) sets. Define an equivalence relation \( \sim \) on \( S \) (i.e., describe precisely what ”\( x \sim y \)” means) such that the equivalence classes for this relation are exactly the sets \( A_1, \ldots, A_n \). (No proofs required. Just give the precise definition for ”\( x \sim y \).”)

Solution: The relation defined by

\[ x \sim y \iff x \text{ and } y \text{ belong to the same set } A_i \]

is an equivalence relation on \( S \) whose equivalence classes are \( A_1, \ldots, A_n \).

2. For the following questions, either give a concrete example with the requested properties, or explain (in one or two sentences) why no such example exists.

(a) A function from \( \mathbb{R} \) to \( \mathbb{R} \) that is increasing, but not surjective.

Solution: \( f(x) = e^x \).

(b) A function from \( \mathbb{R} \) to \( \mathbb{R} \) that has an inverse, but is not injective.

Solution: No such function exists, since if \( f \) has an inverse, then \( f \) is a bijection and hence injective.

(c) A bijection between \( \mathbb{Z} \) (the set of all integers) and \( \mathbb{Z}_{\text{odd}} \) (the set of odd integers).

Solution: \( f(n) = 2n - 1 \).

(d) A bounded subset of \( \mathbb{R} \) that has the same cardinality as \( \mathbb{N} \times \mathbb{N} \).

Solution: Since \( \mathbb{N} \times \mathbb{N} \) is countable (as can be seen by the ”zigzag” argument that shows the countability of \( \mathbb{Q} \)), any countable bounded subset of \( \mathbb{R} \) will do; for example, the set \( \{1/n : n \in \mathbb{N}\} \).

3. Negate the following statements without using words of negation. State your negation in English. Be sure to include any necessary quantifiers, in the correct order. (Here \( f \) is assumed to be a function from \( \mathbb{R} \) to \( \mathbb{R} \).)

(a) If \( x \neq 0 \) and \( y \neq 0 \), then \( f(xy) \neq 0 \).

Solution: [3(a), Logic worksheet]

Symbolic notation: \( (\forall x, y \in \mathbb{R})[x \neq 0 \land y \neq 0 \Rightarrow f(xy) \neq 0] \)

Negation: \( (\exists x, y \in \mathbb{R})[x \neq 0 \land y \neq 0 \land f(xy) = 0] \)

"There exist \( x, y \in \mathbb{R} - \{0\} \) such that \( f(xy) = 0.\"

Or:

"There exist \( x, y \in \mathbb{R} \) such that \( x \neq 0, y \neq 0, \) and \( f(xy) = 0.\"

(b) There exists \( M \in \mathbb{R} \) such that for all \( x \in \mathbb{R} \) there exists \( y > x \) such that \( f(y) \leq M \).

Solution: [4(e) on the Logic worksheet]

Symbolic notation: \( (\exists M \in \mathbb{R}) (\forall x \in \mathbb{R})(\exists y > x)(f(y) \leq M) \)

Negation: \( (\forall M \in \mathbb{R}) (\forall x \in \mathbb{R})(\forall y > x)(f(y) > M) \)

"For all \( M \in \mathbb{R} \) there exists \( x \in \mathbb{R} \) such that for all \( y > x \) we have \( f(y) > M.\"

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5. Using induction, prove that the Fibonacci sequence (defined by \(F_1 = 1, F_2 = 1, \) and \(F_n = F_{n-1} + F_{n-2}\) for \(n \geq 3\)) satisfies \(F_n \geq (3/2)^{n-2}\) for all \(n \in \mathbb{N}\).

Your write-up must include all steps, in the correct logical order, and with appropriate justifications for each step (e.g., “by induction hypothesis,” “by AGM,” “by equation (1)”).

**Solution:**

**Base case:** When \(n = 1\), the left side of (*) is \(F_1 = 1\), and the right side is \((3/2)^{-1} = 2/3\), so (*) holds for \(n = 1\). When \(n = 2\), the left side of (*) is \(F_2 = 1\), and the right side is \((3/2)^{0} = 1\), so both sides are equal and (*) is true for \(n = 2\).

Thus, (*) holds for \(n = 1\) and \(n = 2\).

**Induction step:** Let \(k \geq 2\) be given and suppose (*) is true for all \(n = 1, 2, \ldots, k\). Then

\[
F_{k+1} = F_k + F_{k-1} \quad \text{(by recurrence for } F_n) \\
\geq (3/2)^{k-2} + (3/2)^{k-3} \quad \text{(by strong induction hypothesis with } n = k \text{ and } n = k - 1) \\
= (3/2)^{k-1} ((3/2)^{-1} + (3/2)^{-2}) \quad \text{(by algebra)} \\
= (3/2)^{k-1} \left( \frac{2}{3} + 4 \right) \\
= (3/2)^{k-1} \frac{10}{9} > (3/2)^{k-1}.
\]

Thus, (*) holds for \(n = k + 1\), and the proof of the induction step is complete.

**Conclusion:** By the principle of induction, it follows that (*) is true for all \(n \in \mathbb{N}\).
6. (a) **Using the definition of a limit**, prove that if \( \lim_{n \to \infty} a_n = L \), then \( \lim_{n \to \infty} a_n^2 = L^2 \). (The proof should be done directly from the “\( \epsilon - N \)” definition of the limit of a sequence, and **not** use any of the properties, lemmas, propositions, etc. of limits established in the book, in class, or in the hw problems.)

**Solution:** Suppose \( \lim_{n \to \infty} a_n = L \). Let \( \epsilon > 0 \) be given.

We first apply the definition of convergence with \( \epsilon' = 1 \) to get \( N_1 \in \mathbb{N} \) such that, for \( n \geq N_1 \),

\[
|a_n - L| < 1 \quad 1 \leq |a_n - L| \leq 1 + 2|L| \quad (n \geq N_1).
\]

Next, we apply the definition of convergence with \( \epsilon' = \epsilon/(1 + 2|L|) \) to get

\[
|a_n - L| < \epsilon' = \frac{\epsilon}{1 + 2|L|} \quad (n \geq N_2).
\]

Now let \( N = \max(N_1, N_2) \). Then, for \( n \geq N \), (1) and (2) hold, and so

\[
|a_n^2 - L^2| = |a_n - L| \cdot |a_n + L| < \frac{\epsilon}{1 + 2|L|} (1 + 2|L|) = \epsilon.
\]

Thus, we have shown that \( n \geq N \) implies \( |a_n^2 - L^2| < \epsilon \). Since \( \epsilon > 0 \) was arbitrary, this proves that \( \lim_{n \to \infty} a_n^2 = L^2 \).

(b) **Show by a counterexample, that the converse is not true.** That is, give a specific example of a sequence \( \{a_n\} \) for which \( \lim_{n \to \infty} a_n \) exists, but \( \lim_{n \to \infty} a_n^2 \) does not exist.

**Solution:** Let \( a_n = (-1)^n \). Then \( \lim_{n \to \infty} a_n \) does not exist, but \( \lim_{n \to \infty} a_n^2 = \lim_{n \to \infty} 1 = 1 \).

7. (a) **Give a careful proof** that the sequence defined by \( x_1 = 1 \), and \( x_{n+1} = \sqrt{2x_n} \) for \( n \geq 1 \) converges. (Use induction if necessary. You may use any of the named convergence theorems covered in class; if you do so, state which theorem you are using and be sure to show that the conditions in the theorem are satisfied.)

**Solution:** We use the Monotone Convergence theorem. To prove convergence, we need to show that the sequence is bounded and monotone.

**Boundedness:** We will show by induction that \( (*) \) \( 1 \leq x_n \leq 2 \) for all \( n \in \mathbb{N} \).

In the base case \( n = 1 \), we have \( x_1 = 1 \), so \( (*) \) holds in this case. For the induction step, suppose that \( k \in \mathbb{N} \) and that \( (*) \) holds for \( n = k \). Then, by the recurrence for \( x_{k+1} \), and the induction hypothesis \( 1 \leq x_k \leq 2 \), we have

\[
x_{k+1} = \sqrt{2x_k} \geq \sqrt{2} \geq 1,
\]

\[
x_{k+1} = \sqrt{2x_k} \leq 2 \cdot 2 = 4 \leq 2,
\]

so \( 1 \leq x_{k+1} \leq 2 \). Hence \( (*) \) holds for \( n = k + 1 \), completing the induction step.

By the principle of induction, it follows that \( 1 \leq x_n \leq 2 \) holds for all \( n \in \mathbb{N} \). Thus, the sequence \( \{x_n\} \) is bounded, with 2 as bound.

**Monotonicity:** We will show that the sequence \( \{x_n\} \) is nondecreasing by showing that \( x_{n+1}/x_n \geq 1 \) for all \( n \in \mathbb{N} \). For any \( n \in \mathbb{N} \) we have

\[
\frac{x_{n+1}}{x_n} = \frac{\sqrt{2x_n}}{x_n} = \sqrt{\frac{2}{x_n}} \quad \text{(by recurrence for } x_{n+1} \text{)}
\]

\[
= \sqrt{\frac{2}{x_n}} \quad \text{(by algebra)}
\]

\[
\geq 1 \quad \text{(since } x_n \leq 2 \text{ by } (*)).
\]

so \( x_{n+1}/x_n \geq 1 \), i.e., \( x_{n+1} \geq x_n \), as claimed.

**Conclusion:** Since the sequence \( \{x_n\} \) is bounded and nondecreasing, the Monotone Convergence Theorem implies that it is convergent.
8. For each of the following statements, determine if it is true or false. If it is true, give a brief justification (e.g., by citing an appropriate theorem or property); if it is false, give a specific counterexample.

(a) If the limits \( L = \lim_{n \to \infty} \frac{x_n}{x_{n-1}} \) both exist and \( L \leq M \), then there exists \( N \in \mathbb{N} \) such that \( a_n \leq b_n \) for all \( n \geq N \).

**Solution:** FALSE. Counterexample: \( a_n = 1/n \), \( b_n = 0 \).

(b) If \( \sum_{n=1}^{\infty} a_n \) and \( \sum_{n=1}^{\infty} b_n \) both converge, then \( \sum_{n=1}^{\infty} a_n b_n \) converges.

**Solution:** FALSE. Counterexample: \( a_n = (-1)^n / \sqrt{n} \), \( b_n = (-1)^n / \sqrt{n} \).

(c) If \( \lim_{n \to \infty} (a_{n+1} - a_n) = 0 \), then the sequence \( \{a_n\} \) is bounded.

**Solution:** FALSE. Counterexample: \( a_n = \sqrt{n} \).

(d) If there exists \( M \in \mathbb{R} \) such that for all \( n, m \in \mathbb{N} \), \( |a_n - a_m| < M \), then the sequence \( \{a_n\} \) has a convergent subsequence.

**Solution:** TRUE. The given condition implies that \( |a_n| = |a_n - a_1 + a_1| \leq |a_n - a_1| + |a_1| < M + |a_1| \), so \( \{a_n\} \) is bounded. By the Bolzano-Weierstrass Theorem, it therefore has a convergent subsequence.

9. (a) Find the remainder of 2011\(^{2012} \mod 9 \). We need to find the least nonnegative residue of 2011\(^{2012} \mod 9 \). Now 2011 \equiv 2 + 0 + 1 + 1 = 4 \mod 9, so 2011\(^{2012} \equiv 4^{2012} \mod 9 \). Also, 4\(^3 = 64 \equiv 1 \mod 9 \), and 2012 = 3 \cdot 670 + 2, so

\[
2011^{2012} \equiv 4^{2012} \equiv 4^{3 \cdot 670 + 2} \equiv (4^3)^{670} \cdot 4^2 \equiv 1^{670} \cdot 4^2 \equiv 16 \equiv 7 \mod 9.
\]

(b) Prove that, for any integer \( n \geq 2 \), the number \( n^{100} + 1 \) is composite and find a nontrivial divisor of this number.

**Solution:** This is similar to the proof (given in class) that a number of the form \( 2^n + 1 \) is composite whenever \( n \) is not a power of 2. The idea is to split off the odd part of the exponent 100, write \( n^{100} + 1 = (n^4)^{25} + 1 \), and consider congruences to \( n^4 + 1 \):

\[
n^4 \equiv -1 \mod n^4 + 1,
\]

\[
n^{100} = (n^4)^{25} \equiv (-1)^{25} = -1 \mod n^4 + 1,
\]

\[
n^{100} + 1 \equiv 0 \mod n^4 + 1.
\]

This shows that \( n^4 + 1 \) divides \( n^{100} + 1 \) for all \( n \in \mathbb{N} \). Since for \( n \geq 2 \), \( 1 < n^4 + 1 < n^{100} + 1 \), this proves that \( n^4 + 1 \) is composite for \( n \geq 2 \), with \( n^4 + 1 \) as a nontrivial factor.

(c) Prove that 453 \cdot 347^n + 408^{n+1} is divisible by 5 for all odd positive integers \( n \).

**Solution:** The assertion is equivalent to 453 \cdot 347^n + 408^{n+1} \equiv 0 \mod 5, for all odd \( n \in \mathbb{N} \). Reducing everything modulo 5, we get

\[
453 \cdot 347^n + 408^{n+1} \equiv 3 \cdot 2^n + 3^{n+1} \equiv 3 \cdot 2^n + (-2)^{n+1} \mod 5
\]

For \( n \) odd, we have \((-2)^{n+1} = 2^{n+1} \), so the above becomes \( 3 \cdot 2^n + 2^{n+1} = 5 \cdot 2^n \equiv 0 \mod 5 \), which proves the claim.
10. (a) Prove that \( \log_{10} 2 \) (the base 10 logarithm of 2) is irrational. (Hint: Use a famous theorem.)

**Solution:** We argue by contradiction and suppose that \( \log_{10} 2 \) is rational. Then there exist positive integers \( r \) and \( s \) such that \( \log_{10} 2 = r/s \). By the definition of the base 10 logarithm, this is equivalent to \( 10^{r/s} = 2 \), or \( 10^r = 2^s \). By the Fundamental Theorem of Arithmetic (the “famous theorem” referred to in the problem) such an equation can only hold if the prime factors on the left and right sides, along with their exponents, match. But this forces \( r = 0 \) and \( s = r = 0 \), contradicting the assumption that \( r \) and \( s \) are positive integers. This contradiction proves that \( \log_{10} 2 \) cannot be rational.

(b) Using Euclid’s method, give a careful proof of the infinitude of primes.

**Solution:** We argue by contradiction. Assume there are only finitely many prime numbers, say \( p_1, p_2, \ldots, p_n \). Let \( N = p_1 p_2 \ldots p_n + 1 \). Then \( N \) is an integer \( \geq 2 \), so by the Fundamental Theorem of Arithmetic it can be written as a product of one or more prime numbers. Since, by our assumption, \( p_1, \ldots, p_n \) are all the primes, at least one of these, say \( p_i \), must appear in the factorization of \( N \), and so \( p_i \mid N \). On the other hand, \( p_i \mid p_1 p_2 \ldots p_n = N - 1 \). Since \( d \mid a \) and \( d \mid b \) implies \( d \mid a - b \) (see Problem 2 below), it follows that \( p_i \mid N - (N - 1) = 1 \), which is impossible, since \( p_i \geq 2 \). This contradiction proves the claim.

11. **Extra Credit Questions.**

(a) Determine, with a careful proof, the precise set of functions \( f \) from \( \mathbb{R} \) to \( \mathbb{R} \) for which the following statement holds. Describe this set in as simple a language as possible. (You can use terminology from Chapters 13 and 14.)

For every \( \epsilon > 0 \) and every \( y \in \mathbb{R} \) there exists \( x > y \) such that for all \( \delta > 0 \), \( |x| < \delta \) implies \( |f(x)| < \epsilon \).

**Solution:** To begin with let’s analyze the innermost statement

\[(*) \quad \text{“for all } \delta > 0, \text{ } |x| < \delta \text{ implies } |f(x)| < \epsilon.”\]

Note that this phrase appears after the variables \( \epsilon \) and \( x \), so those variables are fixed once we get to this statement. First, if \((*)\) holds, then the implication “\(|x| < \delta \text{ implies } |f(x)| < \epsilon\)” must be true for all \( \delta > 0 \), and thus, in particular, for \( \delta = |x| + 1 \). With this choice of \( \delta \), the premise “\(|x| < \delta\)” is true, so the conclusion, i.e., “\(|f(x)| < \epsilon\)” must be true as well. Thus, \((*)\) implies

\[(**) \quad |f(x)| < \epsilon.\]

Conversely, if \((**)\) holds, then so does \((*)\), as the conclusion in the implication is satisfied. Thus, \((*)\) and \((**)\) are equivalent, so the given statement can be simplified to:

“For every \( \epsilon > 0 \) and all \( y \in \mathbb{R} \) there exists \( x > y \) such that \(|f(x)| < \epsilon\).”

The latter condition is equivalent to

“For every \( \epsilon > 0 \) and all \( y \in \mathbb{R} \), \( \inf \{|f(x)| : x > y\} < \epsilon \)”

which in turn can be simplified to the limit statement,

\[\lim_{y \to \infty} \inf \{|f(x)| : x > y\} = 0,\]

or, even more concisely,

\[\lim_{y \to \infty} \inf |f(y)| = 0.\]

(b) Let \( S \) be a countable set, and let \( P(S) \) be the power set of \( S \), i.e., the set of all subsets of \( S \). Prove that \( P(S) \) is uncountable.
Solution: We argue by contradiction, using a diagonalization trick similar to that proving the uncountability of the reals: Let $S = \{s_1, s_2, \ldots\}$ be a countable set and suppose $P(S)$ is also countable. Then all subsets of $S$ can be enumerated into a sequence $A_1, A_2, A_3, \ldots$. Now define $A$ to be the set of those elements $s_i$ of $S$ that do not belong the subset with the same index, $A_i$; i.e.,

$$A = \{s_i \in S : s_i \not\in A_i\}.$$ 

Then $A$ is a subset of $S$, but $A \neq A_i$ for any $i$ since, by construction, the element $s_i$ belongs to one of the sets $A$ and $A_i$, but not the other. This contradicts our assumption that the sets $A_1, A_2, \ldots$ represent all subsets of $S$. This contradiction proves that the set of such functions is uncountable.