RULES: Same as always: Staple this sheet to the assignment, do the problems in order, write legibly, show all work, and turn in the assignment in class by the above deadline. Grading: 20 points total, broken down as follows:

- **Presentation/effect:** 5 points
- **Graded problems:** 15 points

**Puzzle of the Week:** Here is another game-type puzzle involving a “magic number” that you can try on a friend. It is not particularly difficult, it is not hard to guess a formula for magic number, and proving that this number has the desired “magic” properties is not that hard either.

**#1 A magic matrix.** Consider the \( n \times n \) matrix obtained by filling the rows of this matrix with the numbers 1, 2, \ldots, \( n^2 \), so that the first row consists of the numbers 1, 2, \ldots, \( n \), the second row of the numbers \( n + 1, n + 2, \ldots, n^2 \), and so on. Now choose \( n \) entries in this matrix such that each row and each column contains exactly one of these entries, and add these \( n \) entries. Prove that, no matter how these \( n \) entries are chosen, for a given \( n \)-value their sum is always the same, and equal to a certain magic number \( M_n \).

For example, in the case \( n = 4 \), here are two possible such choices of these 4 entries:

\[
\begin{bmatrix}
1 & 2 & 3 & 4 \\
5 & 6 & 7 & 8 \\
9 & 10 & 11 & 12 \\
13 & 14 & 15 & 16
\end{bmatrix},
\begin{bmatrix}
1 & 2 & 3 & 4 \\
5 & 6 & 7 & 8 \\
9 & 10 & 11 & 12 \\
13 & 14 & 15 & 16
\end{bmatrix}.
\]

The sum of the boxed entries is 3 + 8 + 9 + 14 = 34 for the matrix on the left, and 2 + 5 + 12 + 15 = 34, for the matrix on the right. Trying out more examples one always obtains 34 as sum of the chosen entries, suggesting that \( M_4 = 34 \) is the “magic number” for the case \( n = 4 \). Your task is to prove this rigorously, for matrices of any dimension.

**Proof practice with the \( \epsilon \)-definition of a limit.** For the following problems, give formal proofs using the definition of a limit. These proofs should be done directly from the “\( \epsilon - N \)” definition of the limit of a sequence, and not use any of the properties, lemmas, propositions, etc. of limits established in the book or in class.

**#2 13.25.** Using the \( \epsilon \)-definition of a limit, prove that \( \lim_{n \to \infty} \sqrt{1 + \frac{1}{n}} = 1 \).

**#3** Using the \( \epsilon \)-definition of a limit, prove that \( \lim_{n \to \infty} \frac{3n^3 + 2n + 1}{n^3 + 1} = 3 \).

**#4 13.11(a).** Using the \( \epsilon \)-definition of a limit, prove that if the limits \( L = \lim_{n \to \infty} a_n \) and \( M = \lim_{n \to \infty} b_n \) both exist and \( L < M \), then there exists \( N \in \mathbb{N} \) such that \( n \geq N \) implies \( a_n < b_n \).

**#5 13.26.** Using the \( \epsilon \)-definition of a limit, prove that if \( \lim_{n \to \infty} a_n = 1 \) and \( a_n > -1 \) for all \( n \in \mathbb{N} \), then \( \lim_{n \to \infty} 1/(1 + a_n) = 1/2 \).

**Applications of Monotone Convergence.** For the following problems, use the Monotone Convergence Theorem. The main task is to show that the given sequence satisfies the two conditions of the theorem (bounded and monotone).

**#6 13.29.** Let \( x_n = \frac{1 + n}{1 + 2n} \). Using the Monotone Convergence Theorem, prove that \( \lim_{n \to \infty} x_n \) exists.

**#7 13.30.** Let \( x_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} \). Using the Monotone Convergence Theorem, prove that \( \lim_{n \to \infty} x_n \) exists. (Hint: Note that \( x_n \) is a sum of \( n \) terms, so \( x_{n+1} \) has one more term than \( x_n \).)