Extending theorems from Recurrences. 3.57.
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3.62 (The December 31 Game).
Recurrences. 3.56.
Fibonacci rationals.
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The following puzzle requires a good dose of clever thinking, along with a small dose of reasoning. (In this problem, induction plays only a very minor auxiliary role; the main task lies in justifying/explaining your write-up/explanation of your answer. Try to be as clear and as convincing as possible!)

Basic Induction Problems. These problems are of one the standard types of problems on the Induction Worksheets I and II, and they should be done using induction, as illustrated on the worksheets.

#1 Extending theorems from 2 to n variables: 3.22. Prove by induction that, for any \( n \in \mathbb{N} \) and any real numbers \( a_1, a_2, \ldots, a_n \), \( \sum_{i=1}^{n} a_i \leq \sum_{i=1}^{n} |a_i| \). (See hint on back of page.)

#2 Summation formulas. 3.28. Find and prove by induction a formula for \( \sum_{i=1}^{n} \frac{1}{n+i} \), where \( n \in \mathbb{N} \).

#3 Representation problems. 3.44. Determine the exact set of natural numbers that can be expressed as the sum of some nonnegative number of 3’s and some nonnegative number of 10’s. (See hint on back of page.)

#4 Inequalities: 3.49(b). Determine the exact set of natural numbers \( n \) for which the inequality \( 2^n \geq (n+1)^2 \) holds. (See hint on back of page.)

#5 Recurrences. 3.56. Let \( a_1, a_2, a_3, \ldots \) be a sequence satisfying \( a_n = 2a_{n-1} + 3a_{n-2} \) for \( n \geq 3 \). (i) Prove that if \( a_1 \) and \( a_2 \) are odd, then \( a_n \) is odd for all \( n \in \mathbb{N} \). (ii) Prove that if \( a_1 = 1 \) and \( a_2 = 1 \), then \( a_n = (1/2)(3^{n-1} - (-1)^n) \) for all \( n \in \mathbb{N} \).

#6 Recurrences. 3.57. Let \( a_1 = 1, a_2 = 1, \) and \( a_n = (1/2)(a_{n-1} + 2/a_{n-2}) \) for all \( n \geq 3 \). Prove that (*) \( 1 \leq a_n \leq 2 \) for \( n \in \mathbb{N} \). (See hint on back of page.)

Fibonacci Problems. The Fibonacci sequence \( 1, 1, 2, 3, 5, 8, 13, 21, \ldots \), defined by \( F_1 = 1, F_2 = 1 \) and \( F_{n+1} = F_n + F_{n-1} \) for \( n \geq 2 \), is one of the most famous mathematical sequences. It has many remarkable properties, is full of surprises, and a near endless source of amazing formulas. These formulas are often hard to discover, and the exercises below are intended to practice such proofs.

#7 Sum of Fibonacci squares. Prove that \( \sum_{i=1}^{n} F_i^2 = F_n F_{n+1} \) for all \( n \in \mathbb{N} \).

#8 Exact formula for \( F_n \). Prove the formula \( F_n = \frac{1}{\sqrt{5}} (\alpha^n - \beta^n) \), where \( \alpha = (1 + \sqrt{5})/2 \) (the golden ratio) and \( \beta = (1 - \sqrt{5})/2 \). (See hint on back of page.)

#9 Fibonacci rationals. Let \( a_1 = 1 \) and for \( n \geq 2 \) define \( a_n = 1 + 1/a_{n-1} \). The numbers \( a_n \) form a sequence of positive rational numbers and hence can be expressed in the form \( a_n = p_n/q_n \), where \( p_n \) and \( q_n \) are positive integers. Guess and prove by induction a formula for these numbers.

Puzzle of the Week. The following puzzle requires a good dose of clever thinking, along with a small dose of induction. A key goal of these problems is to practice proof-writing in situations where you cannot follow a standard template, so focus on the write-up/explanation of your answer. Try to be as clear and as convincing as possible! (In this problem, induction plays only a very minor auxiliary role; the main task lies in justifying/explaining your reasoning.)

#10 3.62 (The December 31 Game). The problem asks you to determine all “winning” positions for this game. See the text for the precise statement of the game. (See back of page for a hint.)

*** Turn page for comments, hints, and advice ***
Hints, Advice, and Comments

**General comments.** All problems in this assignment are exercises in induction; you should do them by induction, and not by other methods. Except for the last problem (a puzzle problem), they all fall into one of the standard types of induction proofs discussed in the first two Induction Worksheets.

Induction proofs (especially, routine type proofs like those in this assignment) provide an ideal setting in which to practice and perfect your proof writing skills. **You should take the write-up seriously and strive for a proof that is as close to perfect as possible in every respect: the logic, the mathematics, the notation, the visual presentation (e.g., display long chains of equations), and the English (e.g., use proper spelling, grammar, and punctuation).** Use the examples on the worksheets as models.

In particular, an induction proof should include the following:

- **A precise statement of the statement/proposition to be proved (i.e., the statement “P(n)”).**
- **The base step.** Usually, this consists of checking $P(n)$ for a single initial $n$-value, but in some applications of strong induction (e.g., problems involving recurrence sequences and representation problems), one needs to check $P(n)$ for two or more consecutive $n$-values before the induction step can be applied.
- **The induction step.** The proof of the induction step is the crux of the argument, and it must be given in full detail, with each of the steps justified (marginal notes like “by inductive hypothesis” or “by algebra” are okay). **Always clearly indicate, at the appropriate place in the induction step, where the induction hypothesis is being used.**
- **Conclusion.** An overall conclusion, e.g.: “By the principle of induction, this proves ...”

**Comments on specific problems.**

- **3.22:** The inequality to prove is an $n$-variable version of the Triangle Inequality, which states that $|x+y| \leq |x|+|y|$ for any real numbers $x$ and $y$. The Triangle Inequality (which you may assume for this problem) is the case $n = 2$ of the asserted inequality; use induction to prove the general case.

- **3.44 and 3.49.** These problems require two tasks: (i) checking the first few cases directly to determine for which of these the property holds and find an appropriate starting point for an induction argument; (ii) using induction (or strong induction) to prove that the given property holds for all $n$ from some point onwards.

- **3.57.** The induction step requires some care; you will need to use properties of inequalities between real numbers (of the type that came up in Chapter 1). Also, keep in mind that the two-sided inequality $1 \leq a_n \leq 2$ is equivalent to the two inequalities $a_n \geq 1$ and $a_n \leq 2$ holding simultaneously. In the induction step you will need to use each of these inequalities at some point, and you should clearly state which of the inequalities you are using, and for which value of the index $n$ you are using it.

- **3.62.** (The December 31 Game.) The back of the text has a small hint. Encode the “winning” positions in this game as $(m, d)$, where $m$ is the month and $d$ is the date of the month. $(12, 31)$ (i.e., December 31) is obviously a winning position, but so is $(11, 30)$ since for this position, the only possible move by your opponent is to increase the month to 12 (i.e., December), and you can then counter by increasing the day to 31, which gets you to December 31. **Working “backwards”, try to determine all winning positions for December, November, October, etc.**

- **Exact formula for $F_n$:** That there is such a simple exact formula for $F_n$ is rather unexpected—even more so since the formula involves the irrational number $\sqrt{5}$, and it is not at all all obvious that it should give a rational number, let alone an integer!

The proof is a rather routine exercise in strong induction, of the same type as the proof of 3.56(b)), though the algebraic calculations required can be a bit messy. Here is a trick to simplify the algebra: Use the fact that $\alpha$ and $\beta$ satisfy the equations $\alpha^2 = \alpha + 1$ and $\beta^2 = \beta + 1.$