Set-theory and logic problems: The first group of problems are some additional problems to practice set-theoretic proofs, and the interpretation of logical statements.

#1 1.50. Let \( f \) be a function from \( \mathbb{R} \) to \( \mathbb{R} \). (a) Prove that, for any sets \( C \subseteq \mathbb{R} \) and \( D \subseteq \mathbb{R} \), \( f(C \cap D) \subseteq f(C) \cap f(D) \).
(b) Give an example (i.e., a specific choice of \( f \) and the sets \( C \) and \( D \)) where equality does not hold in (*) . Explain/justify your answer.
(Hint: \( f(S) \) is defined as \( f(S) = \{ f(s) : s \in S \} \), or equivalently \( f(S) = \{ b \in B : \text{there exists } a \in S \text{ such that } f(a) = b \} \). The latter form is particularly convenient for this problem.)

#2 2.24. Let \( S \) be a set of real numbers (i.e., \( S \subseteq \mathbb{R} \)): Consider the following statements:
(A) There is a number \( M \) such that, for every \( x \) in the set \( S \), \( |x| \leq M \).
(B) For every \( x \) in the set \( S \), there is a number \( M \) such that \( |x| \leq M \).
(a) Which sets satisfy statement (A)? Describe these sets in “simple language” and explain your answer. (There is an extremely simple (essentially one-word) description of these sets; you’ll know it when you get it!)
(b) Which sets satisfy statement (B)? Describe these sets in “simple language” and explain your answer. (Again there is an extremely simple description of these sets.)
(c) Which of the properties implies the other? Explain your answer, and give an example of a set that satisfies one property, but not the other.

#3 2.26. Let \( a \in \mathbb{R} \) and \( f \) be a function from \( \mathbb{R} \) to \( \mathbb{R} \). Consider the following statements:
(A) \((\forall \varepsilon > 0)(\exists \delta > 0)[|x - a| < \delta \implies (|f(x) - f(a)| < \varepsilon)]]\)
(B) \((\exists \delta > 0)(\forall \varepsilon > 0)[|x - a| < \delta \implies (|f(x) - f(a)| < \varepsilon)]]\)
(a) Which of the statements is stronger, i.e., implies the other? Give a clear and convincing explanation for your answer.
(b) Find a function that satisfies one statement, but not the other. Justify your answer (i.e., show why this function satisfies one property, but not the other).
(c) Extra-credit challenge (no collaboration, or help on EC problems!). Determine, again with a clear, convincing argument, the exact set of functions that satisfy (B).

#4 2.28. Consider the equation \( x^4y + ay + x = 0 \).
(a) Show that the following statement is false. “For all \( a, x \in \mathbb{R} \), there is a unique \( y \) such that \( x^4y + ay + x = 0 \).” (Hint: the crucial word here is “unique”.)
(b) Find the set of real numbers \( a \) such that the following statement is true: “For all \( x \in \mathbb{R} \), there is a unique \( y \) such that \( x^4y + ay + x = 0 \).”

Sum/product notations. The following are exercises in properly interpreting and “unwinding” sum/product notations, similar to those on the sum/product notation worksheet. In each case, evaluate the given sum or product for general \( n \in \mathbb{N} \), by writing out the sum/product explicitly, or by using basic manipulations of sums and products and summation formulas from the worksheet. (No need to use induction.) All of the given expressions have a simple general formula involving only elementary functions of \( n \) and the factorial function \( n! \).

#5 \( \sum_{i=1}^{n} n^i \) (This is 3.37 from the text.)
#6 \( \prod_{i=1}^{n} n^i \)
#7 \( \prod_{i=1}^{n} (2i) \)
#8 \( \prod_{i=1}^{n} (2i - 1) \) (Hint: Write out explicitly and fill in the “holes”)
#9 \( \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{1}{j} \) (Hint: There is a very simple formula for this sum!

If you worked with another student or in a small group on this assignment, list the names of all students involved.