Logical statements. The first group of problems are all exercises/drills in working with logical statements. Some involve mathematical definitions, others are ordinary English sentence. For problems asking for negations of statements, express the negation as an English sentence (i.e., without using logical symbols), and without using words of negation. See the Logic Worksheet for examples.

#1 2.4(a)–(f). (Negating mathematical definitions/statements)

#2 2.9. Negate the statement “No slow learners attend this school.” (State the negation of the given statement (among the choices given), and explain why this is the correct negation of the original statement.)

#3 2.10.(a), (c), (d) (Converting English statements into “if ... then” form and negating such statements. State your answers using words, not logical symbols.)

#4 2.23. (Negating a complex mathematical definition.)

#5 Consider the statement “If \( n \) is a perfect number, then \( n \) is even,” where \( n \) is assumed to be an integer. (The definition of “perfect number” doesn’t matter for this problem, but if you are curious, google “odd perfect numbers”; it’s one of the oldest, and most famous open problems in number theory.)
State (a) the converse of this statement; (b) the contrapositive of this statement; (c) the negation of this statement, avoiding using words of negation; (d) an equivalent statement using the phrase “is a necessary condition for”.

Fun problems and puzzles. These problems don’t require any of the formal methods we have developed, just a bit of clever thinking. (See the back of the page for some comments and tips.) None of the problems is particularly difficult, and the point of these problems is not so much on the solution itself, but on the presentation and write-up of the solution. Try to be as clear and as convincing as you can in your write-up. Try to be concise, without skipping any key logical steps.

#6 1.22. (Water/wine puzzle.)

#7 2.11. (Penny/dime/dollar puzzle.)

#8 2.32. (Liars puzzle.)

*** Turn page for comments, hints, and advice ***
Homework HW 2 Advice, Hints, and Comments

• **Know the logical equivalents of common English phrases:** Many problems involve translating statements in English into logical notation and vice versa. See the Logic Handout and p. 29 and 33 (top) of the text for lists of English words or phrases that are commonly used to express a logical implication (⇒) or a universal or existential quantifier (∀, ∃). Familiarize yourself with these words/phrases, and their logical meaning, so that you recognize them they come up. It is impossible to list all such English words, so if you come across a term that is not among those standard terms, just use common sense; for example, ask yourself if it has the same meaning as one of the standard terms.

• **Know how to correctly negate an implication:** One of the most common beginners’ mistakes in proofs is to incorrectly negate an implication. As was pointed out in class and on the Logic Handout, an implication \( P \Rightarrow Q \) is false if and only if \( P \) is true and \( Q \) is false, so the negation of \( P \Rightarrow Q \) is the statement \( P \land \neg Q \). This is essentially the only correct way to negate the implication \( P \Rightarrow Q \). In particular, and this cannot be emphasized enough:

  A negation of an implication \( P \Rightarrow Q \) is never equivalent to another implication involving some combination of \( P, \neg P, Q, \neg Q \).

• **Be aware of implied/hidden quantifiers.** As mentioned in class, statements like “if \( n \) is odd then \( n \) is prime”, or “\( f(x) < f(y) \) whenever \( x < y \)” contain implied/hidden universal quantifiers for the variables \( n \), resp. \( x \) and \( y \). In order to come up with the proper negation of these statements, such implied quantifiers must be made explicit (e.g., “\( \forall n \in \mathbb{Z} \)” resp. “\( (\forall x \in \mathbb{R})(\forall y \in \mathbb{R}) \)”). After negation, these quantifiers turn into existential quantifiers, and these cannot be omitted.

• **Tips on writing up proofs.** The only problems in this assignment requiring full length proofs are the three puzzle problems at the end. These problems are quite different from the more formal proof problems (“even/odd” type proofs and set-theoretic proofs) that we have encountered in HW 1. There isn’t a template you can follow, so you have to come up with the best way to present your argument in each case. **Most of your effort on these problems should go towards the write-up of your argument.** Here are some tips:

  – **Start out with a draft, or outline of your argument, on scratch paper.** Don’t attempt to get the “perfect” proof in one go. Once you have a draft, read it over carefully, ask yourself if it makes logical sense and presents the argument in a clear and convincing manner. Make changes if necessary.

  – **Wait a day or two, and then read over your draft again.** This can help spot errors with your argument that you didn’t notice right way, in the heat of the battle, so to speak.

  – **If possible, have friends or classmates take a look at your proof and provide feedback.** If you are working with other students, read and critique each others’ write-ups.

• **Water/wine puzzle.** The problem is interesting because it has a surprising answer, and also because it has (at least) two very different solutions, a standard algebraic approach, and a short, slick, and elegant alternate approach that requires no algebra at all, and in fact yields a much stronger result! The algebra solution is good enough, but if it piques your interest, try to come up with the slick solution!

• **Penny/dime/dollar puzzle.** This has a short, sweet solution that requires just a small bit of insight. The “right” insight may come to you right away after reading the problem, or after a couple of minutes of thinking it over, or the next day after sleeping it over. Don’t give up too soon; the problem is not hard, and once you see the solution you wonder why didn’t come up with it earlier.

• **Liars’ puzzle.** This can be done with a systematic analysis and elimination of cases.