Worksheet: Epsilonics, II: Infinite Series

Practice Problems

Below are some problems to practice proof-writing skills in the context of infinite series. All of these proofs should be done rigorously, using the “official” definitions of series convergence and results such as Cauchy’s Criterion. None of these proofs is particularly difficult; try to master them all! For additional practice problems, especially of the “prove or find counterexample” variety, see this week’s homework assignment.

A common mistake is to try to “prove” such results by directly manipulating infinite sums \( \sum_{k=1}^{\infty} a_k \). E.g., trying to prove the sum property by writing \( \sum_{k=1}^{\infty} (a_k + b_k) = \sum_{k=1}^{\infty} a_k + \sum_{k=1}^{\infty} b_k \), or the Absolute Convergence Test by writing \( |\sum_{k=1}^{\infty} a_k| \leq \sum_{k=1}^{\infty} |a_k| \), would be totally wrong and would completely miss the point of having precise mathematical definitions of convergence and divergence of series. (With this sort of manipulation one could “prove” all sorts of nonsensical results.)

An infinite sum \( \sum_{k=1}^{\infty} a_k \) has a priori no meaning except as a formal expression. You should NEVER work with infinite sums in your proof. Always work with finite sums, such as the partial sums \( \sum_{k=1}^{n} a_k \), or the sums of finite “chunks” of the series, \( \sum_{k=m+1}^{n} a_k \), that arise in Cauchy’s criterion.

1. Convergence/divergence of particular series:
   
   (a) **Geometric series, convergent case:** Using the definition of convergence of an infinite series, prove that if \( |r| < 1 \), then the geometric series \( \sum_{k=0}^{\infty} r^k \) converges with sum \( 1/(1-r) \). (Hint: Use the formula for the sum of a finite geometric series.)
   
   **Proof:** Let \( s_n = \sum_{k=0}^{n} r^k \) denote the \( n \)-th partial sum of the series. We seek to show that \( \lim_{n \to \infty} s_n = 1/(1-r) \).
   
   By the formula for the sum of a finite geometric series, we have
   
   \[ s_n = \sum_{k=0}^{n} r^k = \frac{1-r^{n+1}}{1-r} = \frac{1}{1-r} - \frac{r}{1-r} r^n. \]
   
   Since \( |r| < 1 \), we have \( \lim_{n \to \infty} r^n = 0 \) (for a formal proof see below). Using the properties of limits, it follows that
   
   \[ \lim_{n \to \infty} s_n = \lim_{n \to \infty} \left( \frac{1}{1-r} - \frac{r}{1-r} r^n \right) \quad \text{(by formula (1))} \]
   
   \[ = \lim_{n \to \infty} \frac{1}{1-r} - \frac{r}{1-r} \lim_{n \to \infty} r^n \quad \text{(by algebraic properties of limits)} \]
   
   \[ = \lim_{n \to \infty} \frac{1}{1-r} - \frac{r}{1-r} 0 \quad \text{(since \( * \) \lim_{n \to \infty} r^n = 0; see below)} \]
   
   \[ = \frac{1}{1-r} \quad \text{(since \( \lim_{n \to \infty} c = c \); see Problem 1(a) on the Epsilonics Worksheet).} \]
   
   Thus, the partial sums \( s_n \) converge to \( 1/(1-r) \). By the definition of convergence of an infinite series, this shows that the series \( \sum_{k=0}^{\infty} r^k \) converges, with sum \( 1/(1-r) \).

   **Proof of (\( * \)):** Let \( a_n = \lfloor r^n \rfloor = \lfloor r^n \rfloor \). Since \( |r| < 1 \), the sequence \( \{a_n\} \) is bounded and monotonically decreasing. By the Monotone Convergence Theorem, it therefore has a limit, say \( L = \lim_{n \to \infty} |r^n| \). To show that \( L = 0 \), we apply the properties of limits: \( L = \lim_{n \to \infty} |r^n+1| = \lim_{n \to \infty} |r| \cdot |r^n| = |r| \lim_{n \to \infty} |r^n| = |r| L \), or \( L(1-|r|) = 0 \). Since \( |r| < 1 \), this implies \( L = 0 \), as desired.

   (b) **Geometric series, divergent case:** Prove that if \( |r| \geq 1 \), then the series \( \sum_{k=0}^{\infty} r^k \) diverges. (Hint: Use an appropriate convergence test.)

   **Proof:** In the case \( |r| \geq 1 \) we have \( |r^n| = |r|^n \geq 1 \), so \( r^n \) does not have limit 0 as \( n \to \infty \). By the \( n \)-th term test, the series \( \sum_{k=0}^{\infty} r^k \) therefore diverges.

   (c) **Divergence of harmonic series:** Prove that the harmonic series \( \sum_{k=1}^{\infty} \frac{1}{k} \) diverges. (Hint: Show that Cauchy’s Criterion for Series with is not satisfied, i.e., prove that it satisfies the negation of Cauchy’s Criterion.)

   **Proof:** Let \( \epsilon = 1/2 \). Then, for any \( N \in \mathbb{N} \), we have
   
   \[ \sum_{k=N+1}^{2N} \frac{1}{k} \geq N \cdot \frac{1}{2N} = \frac{1}{2}. \]
   
   Thus, the harmonic series does not satisfy the Cauchy Criterion (with the choices \( \epsilon = 1/2 \) and \( m = N, n = 2N \)), and hence diverges.

2. General Properties of Series.
(a) Proof of Cauchy Criterion for Series: Prove the Cauchy Criterion for series, using the Cauchy Criterion for sequences.

**Proof:** By definition, convergence of an infinite series is equivalent to convergence of the sequence of its partial sums \( s_n \); by the Cauchy Criterion for sequences, the sequence \( \{s_n\} \) converges if and only if

\[
(\forall \epsilon > 0)(\exists N \in \mathbb{N})(\forall m, n \geq N)[|s_n - s_m| < \epsilon].
\]

Next, observe that we may in (1) assume that \( n \geq m \) since interchanging \( m \) and \( n \) does not affect the quantity \( |s_n - s_m| \); in addition, we may assume \( n \neq m \) since when \( n = m \), \( |s_n - s_n| = 0 \), so the inequality \( |s_n - s_n| < \epsilon \) is satisfied for any \( \epsilon > 0 \). Thus, (1) is equivalent to

\[
(\forall \epsilon > 0)(\exists N \in \mathbb{N})(\forall m > n \geq N)[|s_n - s_m| < \epsilon].
\]

Now, by the definition of \( s_n \) as the \( n \)-th partial sum of the series \( \sum_{k=1}^{\infty} a_k \), we have (under the condition \( n > m \))

\[
s_n - s_m = \sum_{k=1}^{n} a_k - \sum_{k=1}^{m} a_k = \sum_{k=m+1}^{n} a_k.
\]

Substituting this into (2), we can rewrite (2) as

\[
(\forall \epsilon > 0)(\exists N \in \mathbb{N})(\forall n > m \geq N) \left[ \left| \sum_{k=m+1}^{n} a_k \right| < \epsilon \right].
\]

The latter is the Cauchy Criterion for Infinite Series. To summarize, we have the following chain of equivalences:

\[
\sum_{k=1}^{\infty} a_k \text{ converges} \iff \{s_n\} \text{ converges} \iff \{s_n\} \text{ satisfies Cauchy Criterion for sequences, i.e., (1)} \iff \{s_n\} \text{ satisfies (2)} \iff \sum_{k=1}^{\infty} a_k \text{ satisfies (4)} \iff \sum_{k=1}^{\infty} a_k \text{ satisfies Cauchy Criterion for infinite series}
\]

(b) Proof of Absolute Convergence Test: Prove the Absolute Convergence Test using the Comparison Test.

**Proof:** The Absolute Convergence Test says that if \( \sum_{k=1}^{\infty} |a_k| \) converges, then so does \( \sum_{k=1}^{\infty} a_k \). This follows immediately from the Comparison Test using \( b_k = |a_k| \) as the terms in the comparison series.

(c) Series of nonnegative terms: Prove that if a series has (i) only nonnegative terms and (ii) bounded partial sums, then it converges. (Hint: What does the “nonnegative terms” condition mean in terms of the partial sums \( s_n \)?)

**Proof:** Since \( s_{n+1} - s_n = a_{n+1} \), the condition \( a_k \geq 0 \) for all \( k \in \mathbb{N} \) implies that the partial sums \( s_n = \sum_{k=1}^{n} a_k \) satisfy \( s_{n+1} \geq s_n \) for all \( n \in \mathbb{N} \), i.e., they form a monotone sequence. Since we are also given that the partial sums are bounded, the Monotone Convergence Theorem applies and yields that the sequence \( \{s_n\} \) converges. By definition this means that the series \( \sum_{k=1}^{\infty} a_k \) converges.

(d) Proof of the \( n \)-th term test: The \( n \)-th term test says that if \( \sum_{k=1}^{\infty} a_k \) converges, then \( \lim_{n \to \infty} a_n = 0 \). Give a careful proof of the this test, by two methods: (i) using the definition of convergence of infinite series and algebraic properties of limits of sequences; (ii) using the Cauchy Criterion for Series.

**Proof:** Assume \( \sum_{k=1}^{\infty} a_k \) converges. Let \( \epsilon > 0 \) be given. By Cauchy’s Criterion there exists an \( N_1 \in \mathbb{N} \) such that

\[
\left| \sum_{k=m+1}^{n} a_k \right| < \epsilon \quad \text{for all } n, m \in \mathbb{N} \text{ with } n > m \geq N_1.
\]

In particular, applying this with \( n = m + 1 \), (1) yields

\[
|a_n| < \epsilon \quad \text{for all } n \in \mathbb{N} \text{ with } n > N_1.
\]

Now let \( N = N_1 + 1 \). Then \( n > N_1 \) is equivalent to \( n \geq N \), so (2) says that \( n \geq N \) implies \( |a_n| < \epsilon \). By the definition of a limit, this proves \( \lim_{n \to \infty} a_n = 0 \).

(e) Changing finitely many terms in a series does not affect convergence: Suppose \( \sum_{k=1}^{\infty} a_k \) and \( \sum_{k=1}^{\infty} b_k \) are series whose terms differ in at most finitely many places; i.e., there exists an \( N \in \mathbb{N} \) such that \( b_k = a_k \) for all \( k \geq N \). Using the definition of convergence of a series, prove that if \( \sum_{k=1}^{\infty} a_k \) converges, then so does \( \sum_{k=1}^{\infty} b_k \).

**Proof:** This follows by comparing the Cauchy Criterion for the two series. Let \( N_0 \) be such that \( a_k = b_k \) for all \( k \geq N_0 \). Then the sums \( \sum_{k=m+1}^{n} a_k \) and \( \sum_{k=m+1}^{n} b_k \), arising in this criterion are identical provided we choose \( N \) (which is at our disposal) large enough so that \( N \geq N_0 \). (Specifically, given \( \epsilon > 0 \), let \( N_0 \) be an \( N \) that “works” for the series \( \sum_{k=1}^{\infty} a_k \). Then \( N_0 = \max(N_0, N) \) will “work” for the series \( \sum_{k=1}^{\infty} b_k \).