Worksheet: Epsilonics, II: Infinite Series

Sample Proof: Sum Property for Infinite Series

Many properties and results about series can be deduced from corresponding properties/results for sequences. The following is an example of a proof of this type.

**Claim:** If $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ converge with sums $A$ and $B$ respectively, then so does $\sum_{k=1}^{\infty} (a_k + b_k)$, with sum $A + B$.

**Proof.** Suppose $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ converge to $A$ and $B$, respectively.

By the definition of convergence of series, this means that the corresponding partial sums

$$s_n = \sum_{k=1}^{n} a_k, \quad t_n = \sum_{k=1}^{n} b_k$$

converge with limits

$$\lim_{n \to \infty} s_n = A, \quad \lim_{n \to \infty} t_n = B.$$

By the sum property of limits of sequences, it follows that

$$\lim_{n \to \infty} (s_n + t_n) = \lim_{n \to \infty} s_n + \lim_{n \to \infty} t_n = A + B.$$

Since

$$s_n + t_n = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k = \sum_{k=1}^{n} (a_k + b_k)$$

this proves that the series $\sum_{k=1}^{\infty} (a_k + b_k)$, converges, with sum $A + B$, proving our claim.

Sample Proof: Comparison Test

One of the most useful direct methods of proof of convergence (and divergence) of series is the Cauchy Criterion for Series. This example illustrates such an application. Here Cauchy’s Criterion is applied twice, first to the comparison series that is assumed to be convergent and then to the series whose convergence one seeks to prove.

**Claim:** Suppose that $|a_k| \leq b_k$ for all $k \in \mathbb{N}$ and $\sum_{k=1}^{\infty} b_k$ converges. Then $\sum_{k=1}^{\infty} a_k$ converges as well.

**Proof.** Suppose the series $\sum_{k=1}^{\infty} b_k$ converges, and that $|a_k| \leq b_k$ for all $k \in \mathbb{N}$.

Let $\epsilon > 0$ be given.

Since $\sum_{k=1}^{\infty} b_k$ converges, this series satisfies Cauchy’s Criterion, so there exists an $N \in \mathbb{N}$ such that, for all $n, m \in \mathbb{N}$ with $n > m \geq N$,

$$\sum_{k=m+1}^{n} b_k < \epsilon \quad \text{(1)}$$

Let $n > m \geq N$. Then

$$\left| \sum_{k=m+1}^{n} a_k \right| \leq \sum_{k=m+1}^{n} |a_k| \quad \text{(by the $k$-variable triangle inequality, } |x_1 + \cdots + x_k| \leq |x_1| + \cdots + |x_k|)$$

$$\leq \sum_{k=m+1}^{n} b_k \quad \text{(by the assumption } |a_k| \leq b_k)$$

$$< \epsilon \quad \text{(by (1))}$$

Hence, $n > m \geq N$ implies $\left| \sum_{k=m+1}^{n} a_k \right| < \epsilon$. Thus Cauchy’s Criterion holds for the series $\sum_{k=1}^{\infty} a_k$, and this series is therefore convergent.
Practice Problems

Below are some problems to practice proof-writing skills in the context of infinite series. All of these proofs should be done rigorously, using the “official” definitions of series convergence and results such as Cauchy’s Criterion. None of these proofs is particularly difficult; try to master them all! For additional practice problems, especially of the “prove or find counterexample” variety, see this week’s homework assignment.

A common mistake is to try to “prove” such results by directly manipulating \( \sum_{k=1}^{\infty} a_k \). E.g., trying to prove the sum property by writing \( \sum_{k=1}^{\infty} (a_k + b_k) = \sum_{k=1}^{\infty} a_k + \sum_{k=1}^{\infty} b_k \), or the Absolute Convergence Test by writing \( \sum_{k=1}^{\infty} |a_k| \leq \sum_{k=1}^{\infty} |a_k| \), would be totally wrong and would completely miss the point of having precise mathematical definitions of convergence and divergence of series. (With this sort of manipulation one could “prove” all sorts of nonsensical results.)

An infinite sum \( \sum_{k=1}^{\infty} a_k \) has a priori no meaning except as a formal expression. You should NEVER work with infinite sums in your proof. Always work with finite sums, such as the partial sums \( \sum_{k=1}^{n} a_k \), or the sums of finite “chunks” of the series, \( \sum_{k=m+1}^{n} a_k \), that arise in Cauchy’s criterion.

1. Convergence/divergence of particular series:
   
   (a) **Geometric series, convergent case:** Using the definition of convergence of an infinite series, prove that if \(|r| < 1\), then the geometric series \( \sum_{k=0}^{\infty} r^k \) converges with sum \( 1/(1 - r) \). (Hint: Use the formula for the sum of a finite geometric series.)

   (b) **Geometric series, divergent case:** Prove that if \(|r| \geq 1\), then the series \( \sum_{k=0}^{\infty} r^k \) diverges. (Hint: Use an appropriate convergence test.)

   (c) **Divergence of harmonic series:** Prove that the harmonic series \( \sum_{k=1}^{\infty} \frac{1}{k} \) diverges. (Hint: Show that Cauchy’s Criterion is not satisfied, i.e., prove that it satisfies the negation of Cauchy’s Criterion.)

2. General Properties of Series.

   (a) **Proof of Cauchy Criterion for Series:** Prove the Cauchy Criterion for series, using the Cauchy Criterion for sequences.

   (b) **Proof of Absolute Convergence Test:** Prove the Absolute Convergence Test using the Comparison Test.

   (c) **Series of nonnegative terms:** Prove that if a series has (i) only nonnegative terms and (ii) bounded partial sums, then it converges. (Hint: What does the “nonnegative terms” condition mean in terms of the partial sums \( s_n \)?)

   (d) **Proof of the n-th term test:** The n-th term test says that if \( \sum_{k=1}^{\infty} a_k \) converges, then \( \lim_{n \to \infty} a_n = 0 \). Give a careful proof of this test, by two methods: (i) using the definition of convergence of infinite series and algebraic properties of limits of sequences; (ii) using the Cauchy Criterion for Series.

   (e) **Changing finitely many terms in a series does not affect convergence:** Suppose \( \sum_{k=1}^{\infty} a_k \) and \( \sum_{k=1}^{\infty} b_k \) are series whose terms differ in at most finitely many places; i.e., there exists an \( N \in \mathbb{N} \) such that \( b_k = a_k \) for all \( k \geq N \). Using the definition of convergence of a series, prove that if \( \sum_{k=1}^{\infty} a_k \) converges, then so does \( \sum_{k=1}^{\infty} b_k \).