Cardinality: Countable and Uncountable Sets

- **Tool: Bijections.**
  - **Bijection from of a set S:** Let A and B be sets. A bijection from A to B is a function $f : A \to B$ that is both injective and surjective.
  - **Some properties of bijections:**
    - **Inverse functions:** The inverse function of a bijection is a bijection.
    - **Compositions:** The composition of bijections is a bijection.
- **Definitions.**
  - **Cardinality:** Two sets A and B are said to have the **same cardinality** if there exists a bijection from A to B.
  - **Finite and infinite sets:** A set is called **finite** if it is empty or has the same cardinality as the set $\{1, 2, \dots, n\}$ for some $n \in \mathbb{N}$; it is called **infinite** otherwise.
  - **Countable sets:**[1] A set is called **countable** (or **countably infinite**) if it has the same cardinality as $\mathbb{N}$. Equivalently, a set A is countable if it can be **enumerated** in a sequence, i.e., if all of its elements can be listed as a sequence $a_1, a_2, \dots$.A set is called **uncountable** if it is infinite and not countable.
- **Two famous results with memorable proofs.**

  The following are key results in set theory, and their proofs are among the most famous and memorable in all of mathematics. You should know both the results and the proofs.
  - **The rational numbers are countable.**
    Proof idea: "Zigzag method". Arrange the rationals in matrix and enumerate by traversing the matrix in zigzag fashion. (See text, p. 90).
  - **The real numbers are uncountable.** Similarly, any interval $[a, b]$ with $a < b$ is uncountable.
    Proof idea: "Cantor's diagonalization method". Assuming the reals are countable, the decimal expansions of all real numbers in $[0, 1)$ can be put in a matrix with countably many rows and columns. Use the digits in the diagonal to construct a real number in $[0, 1)$ not accounted for, thus obtaining a contradiction. (See text, p. 266.)
- **Some general results on countable and uncountable sets.**
  - **An infinite subset of a countable set is countable; a superset of an uncountable set is uncountable.**
    Proof idea: For the first statement, assume A is a countable set and $B \subseteq A$ an infinite subset of A. Enumerate A as a sequence $a_1, a_2, \dots$. The subsequence consisting of those elements that belong to B then is an enumeration of B. The second statement follows by contraposition. (A "superset" is the opposite of the subset relation, i.e., B is a superset of A if $A \subseteq B$).
  - **A finite or countable union of countable sets countable.** In other words, if $A_1, A_2, \dots, A_k$ are each countable, then so is $A_1 \cup A_2 \cup \dots A_k$, and the same holds for unions of countably many countable sets: If $A_1, A_2, A_3, \dots$ are each countable, then so is the union $A_1 \cup A_2 \cup A_c \cup \cdots$.
    Proof idea: For two countable sets A and B, enumerate the sets as $A = \{a_1, a_2, a_3, \dots\}$ and $B = \{b_1, b_2, b_3, \dots\}$, and "interlace" these enumerations to get an enumeration of the union: $A \cup B = \{a_1, b_1, a_2, b_2, a_3, b_3, \dots\}$. The same method works for unions of finitely many countable sets $A_1, \dots, A_k$.

    For unions of *countably* many sets $A_1, A_2, A_3, \dots$, use the "zigzag" trick: Enumerate each A_i as $\{a_{i1}, a_{i2}, \dots\}$, arrange these elements in an infinite matrix and traverse this matrix in "zigzag" fashion to get an enumeration of all elements a_{ij}, $i \in \mathbb{N}$, $j \in \mathbb{N}$.
  - **The cartesian product of finitely many countable sets is countable.** In other words, if $A_1, \dots, A_k$ are countable, then so is $A_1 \times \cdots \times A_k$.
    Proof idea: For the case of two countable sets A and B, enumerate these sets as $A = \{a_1, a_2, \dots\}$ and $B = \{b_1, b_2, \dots\}$, arrange the elements (a_i, b_j) of $A \times B$ in an infinite matrix and use a "zigzag" argument to traverse this matrix and obtain an enumeration of all matrix elements. To extend this result to the product of k countable sets $A_1 \times \cdots \times A_k$, use induction on k.
    Remark: While unions of countably many countable sets are countable, cartesian products involving countably many countable (or even finite) "factors" need not be countable.

[1] Some texts (e.g., Rosen's "Discrete Mathematics") use the term "countable" in the sense of "finite or countably infinite". We use here the convention of the D'Angelo/West text where the term "countable" is reserved for infinite sets.