1. Consider the two planes given by
\[ 3x + 4y = 1, \quad x + y - z = 3. \]

(i) Find the angle of intersection of these planes. (You can leave this angle in the form of an inverse trig function, e.g., \( \sin^{-1}(\ldots) \) or \( \cos^{-1}(\ldots) \).)

(ii) Find parametric equations for the line of intersection of the planes.

(iii) Find the equation of a plane through the origin that intersects with both planes at a right angle.

2. Let \( T(x, y, z) \) denote the temperature at a point \((x, y, z)\) in space. Suppose that \( T(1, 2, 3) = 32 \), and \( \nabla T(1, 2, 3) = (3, 4, 5) \). Given this information, answer the following questions.

(i) Estimate the value of \( T(1.1, 2.1, 2.9) \).

(ii) Find a linear approximation \( L(x, y, z) \) to the temperature function \( T(x, y, z) \) near the point \((1, 2, 3)\).

(iii) For each of the following parts, find an appropriate unit direction vector, or explain why no such direction vector exist. Use the back of the page for work if needed.)

(a) A direction in which the initial rate of change in temperature is 3:

(b) A direction in which the initial rate of change in temperature is 0:

(c) A direction in which the initial rate of change in temperature is −5:

(d) A direction in which the initial rate of change in temperature is 10:

3. Consider the problem of finding the maximal value of the function \( f(x_1, x_2, \ldots, x_n) = x_1 + 2x_2 + 3x_3 + \cdots + nx_n \) among all vectors \( \vec{x} = (x_1, x_2, \ldots, x_n) \) in \( \mathbb{R}^n \) of norm 1 (i.e., vectors satisfying \( |\vec{x}| = 1 \)).

(i) Set up this problem as a Lagrange multiplier problem, by specifying:

(a) The function to be optimized.

(b) The constraint equation.

(c) The system of equations to be solved.

(Write out all formulas and equations explicitly in terms of the coordinates \( x_1, \ldots, x_n \), not in terms of vectors \( \vec{x} \).

(ii) For this part only, assume \( n = 4 \). In this case (i.e., when \( n = 4 \)) solve the system of equations found in the first part explicitly and use this solution to answer the original question. That is, find the maximal value of \( f(x_1, x_2, x_3, x_4) = x_1 + 2x_2 + 3x_3 + 4x_4 \) among all vectors \( \vec{x} \) of norm 1 using the Lagrange multiplier method. (Use back of page for additional work if needed.)

(iii) For this part, the dimension \( n \) is again arbitrary. Find a short alternate solution to the problem in the case of general dimension \( n \) that does not use Lagrange multipliers or similar optimization techniques from calculus and which requires only a minimal amount of computation. (The solution must be mathematically rigorous and completely justified (e.g., by citing appropriate theorems or properties); no credit will be given for guesswork. The answer may involve a summation, but it must be in “calculator-ready” form: Given a particular \( n \)-value (such as \( n = 4 \) or \( n = 241 \)), performing the summation should yield an explicit numerical value.)

4. The following problems are independent of each other.

(i) Find the volume of the region that lies under the surface \( z = xy \) and above the triangle with vertices \((0, 0), (4, 0), \) and \((0, 2)\).

(ii) Find the mass and the center of mass of the solid hemisphere of radius 1 that is centered at the origin and lies above the \( xy \)-plane, if the mass density function at any point is proportional to the square of the distance of this point to the origin.
5. Let \( \vec{F} \) denote the vector field in \( \mathbb{R}^3 \) defined by \( \vec{F}(x, y, z) = (x + y, y - x, 1) \). Let \( E \) be the solid bounded above by the surface \( z = 1 - \sqrt{x^2 + y^2} \) and below by the \( xy \)-plane, and let \( S \) denote the complete boundary surface of \( E \) oriented with outward normal vectors. Thus \( S \) consists of a “dome” part, \( D \), given by the equation \( z = 1 - \sqrt{x^2 + y^2}, \ z \geq 0 \), and a “floor” part, \( F \), which is a unit disk in the \( xy \)-plane.

(i) Find the flux of \( \vec{F} \) through \( S \) by applying the Divergence Theorem.

(ii) Compute the flux of \( \vec{F} \) through \( S \) directly by computing separately surface integrals \( \int\int_D \vec{F} \cdot d\vec{S} \) and \( \int\int_F \vec{F} \cdot d\vec{S} \) over the “dome” portion \( D \) of \( S \) (oriented with upward pointing normal vector) and the “floor” portion \( F \) of \( S \) (oriented with downward pointing normal vector).

6. Let \( f_p \) and \( \vec{F}_p \) be functions on \( \mathbb{R}^n \) defined by

\[ f_p(\vec{r}) = r^p, \quad \vec{F}_p(\vec{r}) = r^p \vec{r}, \]

where \( \vec{r} = (x_1, \ldots, x_n), r = |\vec{r}|, \) and \( p \) is an arbitrary real number.

Calculate the following quantities, and express your answer in terms of \( \vec{r} \) and/or \( r \) and the parameter \( p \) and dimension \( n \) (but not the coordinates \( x_1, x_2, \ldots \)). Show all work.

(i) Find the gradient of \( f_p \).

(ii) Find the divergence of \( \vec{F}_p \). For which value(s) of \( p \) is the divergence 0?

(iii) For the case \( p \neq -1 \) find a potential for \( \vec{F}_p \). Justify your answer. (Hint: This requires only minimal calculations.)

7. In this problem \( \vec{F} \) denotes the vector field in \( \mathbb{R}^3 \) defined by

\[ \vec{F}(x, y, z) = (x^2, -3z(\sin x)^2, 3y(\cos x)^2). \]

(i) Determine, with proof, whether the vector field \( \vec{F} \) is conservative. Show work!

(ii) Evaluate the line integral \( \oint_C \vec{F} \cdot \vec{T} \, ds \), where \( C \) is the boundary of the square \( x = 1, 0 \leq y \leq 1, 0 \leq z \leq 1 \), traversed in counterclockwise direction (see picture).

8. The following problems are independent of each other.

(i) Express \( \vec{B}' \), the derivative of the binormal vector \( \vec{B} = \vec{B}(t) \), in the TNB frame, i.e., in the form \( \vec{B}' = \ldots \vec{T} + \ldots \vec{N} + \ldots \vec{B} \), with appropriate (scalar) expressions in place of the dots. In particular, determine which (if any) of the three coefficients are equal to 0. Show all work, and justify every step in your argument.

(ii) Consider an object moving at a speed \( v(t) \) along a path \( \vec{r}(t) \). Let \( \theta(t) \) denote the angle between the tangent vector to this path and the position vector at time \( t \).

Let \( d(t) \) denote the distance of the object to the origin at time \( t \). Find a formula for \( d'(t) \), the rate of change of the distance to the origin, in terms of the quantities \( v(t) \) and \( \theta(t) \). The formula should only involve \( v(t) \) and \( \theta(t) \), and not other quantities, such as \( \vec{r}(t), \vec{v}(t), \vec{T}(t) \), etc. Show all work, and justify every step in your argument! (Assume \( \vec{r}(t) \) is differentiable with nonzero derivative.)

9. Definitions and proofs. For each part give the requested formal definition or proof. Be sure to use correct mathematical terminology and notation (e.g., distinguish between vectors and scalars), and clearly define/explain any notation arising in your formula.

(i) Definition of linear combinations and linear functions.

i. Give a precise definition of a linear combination of vectors in \( \mathbb{R}^n \).

ii. Give a precise definition of a linear function \( f \) from \( \mathbb{R}^n \) to \( \mathbb{R}^m \) using the concept of linear combinations. (There are several ways of defining a linear function, but this problem specifically asks for a definition in terms of linear combinations. Be sure to use correct notation and include the appropriate quantifiers (i.e., words like “for any”, “for all”, “for some”).)
(ii) **Definition of differentiability.** Let \( f \) be a function from \( \mathbb{R}^m \) to \( \mathbb{R}^n \), and let \( \vec{x}_0 \) be a point in \( \mathbb{R}^m \). Complete the following definition, by filling in the blanks. (Be sure to use correct notation, and clearly distinguish between vectors and scalars. The definition should involve an appropriate limit.)

\[
f \text{is said to be differentiable at } \vec{x}_0 \text{ if there exists a } \nabla f(\vec{x}_0)
\]

such that

(iii) **Limit definition of divergence.** Give a precise definition of the divergence \( \text{div} \vec{F}(\vec{r}_0) \) of a vector field \( \vec{F} \) in \( \mathbb{R}^3 \) at a given point \( \vec{r}_0 \), as a limit of certain surface integrals. Your answer should be in a form \( \text{div} \vec{F}(\vec{r}_0) = \lim_{\star \to \star} \star \int \star dS \), with appropriate expressions in place of the asterisks. Be sure to define/explain any notation arising in your formula.

(iv) **Vectors in \( \mathbb{R}^n \).** Let \( \vec{x} \) and \( \vec{y} \) be vectors in \( \mathbb{R}^n \). Using the definition of a norm in \( \mathbb{R}^n \) and the properties of norms and dot products for vectors in \( \mathbb{R}^n \), give a careful, step-by-step proof of the following statement:

If \( \vec{x} \) and \( \vec{y} \) are vectors in \( \mathbb{R}^n \) such that \( |\vec{x} + \vec{y}| = |\vec{x} - \vec{y}| \), then \( \vec{x} \) and \( \vec{y} \) are orthogonal.

Be sure to clearly justify each step, e.g., by citing the specific property used. Make sure to use correct notation, and clearly distinguish between vectors and scalars and between various types of multiplication such as dot product, cross product, scalar multiplication.

10. **Bonus question:** Why does the cross product only make sense in \( \mathbb{R}^3 \)? In contrast to other operations with vectors such as addition, multiplication by scalars, dot product, etc., which generalize in an obvious manner to vectors in \( \mathbb{R}^n \), the cross product of two vectors makes only sense in \( \mathbb{R}^3 \), and not in lower-dimensional spaces (\( \mathbb{R} \) and \( \mathbb{R}^2 \)) or higher-dimensional spaces (\( \mathbb{R}^n \) with \( n \geq 4 \)). Explain, as clearly and convincingly as you can, why this is so. (The extra credit will depend on the quality and persuasiveness of the write-up.)