1. Let $L_1$ and $L_2$ denote the lines with parametric equations given by

\[(L_1) \quad \frac{x-1}{-3} = \frac{y-2}{4} = \frac{z+6}{1},\]
\[(L_2) \quad \frac{x-3}{2} = \frac{y}{-2} = \frac{z+5}{1}.\]

(a) Find parametric equations for the two lines.

\[L_1: \quad x = 1 - 3t, \quad y = 2 + 4t, \quad z = -6 + t\]
\[L_2: \quad x = 3 + 2s, \quad y = -2s, \quad z = -5 + s\]

(b) Determine if the lines $L_1$ and $L_2$ are parallel, skew, or intersect at a single point. In the latter case, find the intersection point.

Direction vectors: $\vec{v}_1 = \langle -3, 4, 1 \rangle$, $\vec{v}_2 = \langle 2, -2, 1 \rangle$

The vectors are not parallel, so the lines are skew or intersect.

To determine which, equate $x, y, z$ values in parametric equations and solve:

1. $1 - 3t = 3 + 2s \quad (1)$
2. $2 + 4t = -2s \quad (2)$
3. $-6 + t = -5 + s \quad (3)$

From (3), $s = t + 1$

Plug in (2): $2 + 4t = -2(t + 1) \Rightarrow 4t = 0$

Plug in (3): $-6 + t = -5 + (t + 1) \Rightarrow 6 = 0$

Lines do not intersect at a single point.

(c) Find a linear equation of a plane that is parallel to the lines $L_1$ and $L_2$ and which passes through the origin.

Normal to plane must be perpendicular to both direction vectors $\vec{v}_1$ and $\vec{v}_2$, so we can take $\vec{n} = \vec{v}_1 \times \vec{v}_2$ as normal vector:

\[\vec{n} = \langle -3, 4, 1 \rangle \times \langle 2, -2, 1 \rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 & 4 & 1 \\ 2 & -2 & 1 \end{vmatrix} = \langle 6, 5, -2 \rangle\]

Equation of plane with normal $\vec{n}$ and through point $(0,0,0)$:

\[\vec{n} \cdot (\vec{r} - \vec{r}_0) = 0 \quad \langle 6, 5, -2 \rangle \cdot \langle x, y, z \rangle = 0 \quad 6x + 5y - 2z = 0\]
2. The following problems are independent of each other.

(a) A plane flies along a circular path with radius 4. Its speed at time $t$ is given by $2t$. Express the acceleration of the plane at time $t = 3$ as a linear combination of the vectors $\vec{T}, \vec{N}, \vec{B}$ at this time, i.e., in the form $\vec{a} = \ldots \vec{T} + \ldots \vec{N} + \ldots \vec{B}$, with appropriate coefficients in place of the 3 blanks. (Note that the 3 coefficients should be concrete numbers, not functions of $t$.)

$$\vec{a} = a \vec{T} + a \vec{N} + 0 \vec{B}$$

$$\vec{a} = \vec{v}' = (2t)' = 2 \vec{v}$$

$$a = kv^2 = \frac{1}{4} \cdot (2 \cdot 3)^2 = 9$$

$$\vec{a} = 2 \vec{T} + 9 \vec{N} + 0 \vec{B}$$

(b) Determine all pairs of nonzero vectors $\vec{a}$ and $\vec{b}$ that satisfy $\text{proj}_a \vec{b} = \text{proj}_b \vec{a}$. Explain your reasoning. The answer should be a simple geometric description of all such pairs $\vec{a}$ and $\vec{b}$.

$$\begin{cases}
\text{proj}_a \vec{b} = \text{proj}_b \vec{a} \iff \frac{\vec{a} \cdot \vec{b}}{||\vec{a}||^2} \vec{a} = \frac{\vec{a} \cdot \vec{b}}{||\vec{b}||^2} \vec{b} \\
\iff \begin{cases}
either \frac{\vec{a} \cdot \vec{b}}{||\vec{a}||^2} = 0 \\
or \frac{\vec{a}^2}{||\vec{a}||^2} = \frac{\vec{b}}{||\vec{b}||^2}
\end{cases} \iff \begin{cases}
\vec{a} \perp \vec{b} \quad \text{(Case 1)} \\
\vec{a} = \vec{b} \quad \text{(Case 2)}
\end{cases}
\end{cases}$$

(c) Given a plane determined by three points $A, B, C$, and a point $D$ outside this plane, express the distance between the point $D$ and the plane in terms of cross and/or dot products and/or norms of the vectors $\vec{AB}, \vec{AC}, \vec{AD}$. (The formula should not involve any other vectors, and it should not involve components of vectors. You must clearly show how you arrived at this formula; a picture may be very helpful!)

Distance = $\frac{|\text{scalar projection of } \vec{AD} \text{ onto normal}|}{||\vec{n}||}$

$$= \frac{|\vec{AD} \cdot \vec{n}|}{||\vec{n}||} = \frac{|\vec{AD} \cdot (\vec{AB} \times \vec{AC})|}{||\vec{AB} \times \vec{AC}||}$$
3. **Multiple choice**: circle the correct answer among the choices given. Below \( \vec{r} = \vec{r}(t) \) is the position at time \( t \) of a particle moving in three-dimensional space, \( \vec{v} = \vec{v}(t) \) denotes its velocity, \( v = v(t) \) its speed, \( \vec{a} = \vec{a}(t) \) its acceleration, and \( \vec{T} = \vec{T}(t) \) the unit tangent vector.

(a) The vector \( \vec{r}(t) \) is
   (a) always perpendicular to \( \vec{a}(t) \).
   (b) always parallel to \( \vec{a}(t) \).
   (c) always perpendicular to \( \vec{T}(t) \).
   (d) always parallel to \( \vec{T}(t) \).
   (e) none of the above

(b) The curvature \( \kappa \) is given by
   (a) \( \kappa = \nu' \)  (b) \( \kappa = |\vec{a}|v^2 \)  (c) \( \kappa = \frac{|\vec{T}'|}{|\vec{T}|} \)  (d) \( \kappa = \frac{|\vec{T}'|}{v} \)  (e) none of the above

(c) The **normal** component of the acceleration, \( a_N \), is given by
   (a) \( \nu' \)  (b) \( \kappa \nu' \)  (c) \( \kappa \nu^2 \)  (d) \( \frac{|\vec{T}'|}{v} \)  (d) none of the above

(d) The **tangential** component of the acceleration, \( a_T \), is given by
   (a) \( \frac{\vec{r}' \times \vec{r}''}{|\vec{r}'|} \)  (b) \( \vec{r}' \cdot \vec{r}'' \)  (c) \( \frac{|\vec{r}' \times \vec{r}''|}{|\vec{r}'|^3} \)  (d) \( \frac{|\vec{r}' \cdot \vec{r}''|}{|\vec{r}''|^3} \)
   (e) None of the above.

(e) Assume (for this part only) that \( |\vec{r}(t)| \) is constant. Which of the following conclusions can one draw from this assumption?
   (a) \( \vec{r}(t) \) is perpendicular to \( \vec{a}(t) \), but not necessarily perpendicular or parallel to \( \vec{v}(t) \)
   (b) \( \vec{r}(t) \) is perpendicular to \( \vec{v}(t) \), but not necessarily perpendicular or parallel to \( \vec{a}(t) \)
   (c) \( \vec{r}(t) \) is perpendicular to both \( \vec{v}(t) \) and \( \vec{a}(t) \)
   (d) \( \vec{r}(t) \) is perpendicular to \( \vec{v}(t) \) and parallel to \( \vec{a}(t) \)
   (e) \( \vec{r}(t) \) is perpendicular to \( \vec{a}(t) \) and parallel to \( \vec{v}(t) \)

\* As proved in class, \( |\vec{r}(t)| \) constant \( \iff \vec{r} \perp \vec{r}' \iff \vec{r} \perp \vec{r}' \iff \vec{r} \perp \vec{v} \)

But \( \vec{a} \) can have arbitrary direction.
4. Consider the function \( f(x, y) = \sqrt{y-x} \).

(a) What is the domain of \( f \)? Express the domain as a set using proper set-theoretic notation (i.e., in the form \( \{ \ldots \mid \ldots \} \)). No need to sketch the domain.

\[
D = \{ (x, y) \mid y \geq x^2 \}
\]

(b) What is the graph of \( f \)? Express the graph as a set using proper set-theoretic notation. No need to sketch the graph.

\[
G = \left\{ (x, y, z) \mid y \geq x, \quad z = \sqrt{y-x} \right\}
\]

or

\[
G = \left\{ (x, y, f(x, y)) \mid y \geq x^2 \right\}
\]

(c) Sketch the level curve of \( f \) that passes through the point \((1, 5)\).

Level curve: \( \sqrt{y-x} = k \)

\[
\begin{align*}
\sqrt{y-x} &= k \\
y-x &= k^2 \\
y &= x + k^2
\end{align*}
\]

Lines with slope 1

(d) Find \( f_x(1, 5) \) and \( f_{xy}(1, 5) \).

\[
\begin{align*}
f_x &= -\frac{1}{2} (y-x)^{-\frac{1}{2}} \\
f_x (1, 5) &= -\frac{1}{2} (5-1)^{-\frac{1}{2}} = \frac{1}{\sqrt{4}} = \frac{1}{2} \\
f_{xy} &= (-\frac{1}{2})(-\frac{1}{2})(y-x)^{-\frac{3}{2}} \\
f_{xy} (1, 5) &= \frac{1}{4} (5-1)^{-\frac{3}{2}} = \frac{1}{32}
\end{align*}
\]
5. Let \( \vec{r}(t) \) denote the position of a moving particle at time \( t \), and let \( \vec{v}(t) \) be its velocity at time \( t \). Let \( f(t) \) be the distance of the particle to the origin at time \( t \). Evaluate \( f'(t) \) (i.e., the derivative of the function \( f(t) \)) and express it in terms of \( \vec{r}(t) \) and \( \vec{v}(t) \) (but no other quantities!), simplifying as much as possible.

It is essential that you explain all steps of your work; e.g., if you drop a term, state why you can do that; if you use a property of cross/dot products or derivatives, say so and state which property (e.g., “by chain rule”, “by commutativity”, etc.).

\[
f(t) = |\vec{r}(t)|
\]

\[
f'(t) = \frac{d}{dt} |\vec{r}(t)|
\]

\[
= \frac{d}{dt} \left( \vec{r} \cdot \vec{r} \right)^{\frac{1}{2}} \quad \text{(norm formula)}
\]

\[
= \frac{1}{2} \left( \vec{r} \cdot \vec{r} \right)^{-\frac{1}{2}} \left( \vec{r} \cdot \vec{r} + \vec{r} \cdot \vec{r} \right) \quad \text{(chain/product rule for dot product)}
\]

\[
= \frac{1}{2} \vec{r} \cdot \vec{r} \quad \text{(commutative)}
\]

\[
= \left( \vec{r} \cdot \vec{r} \right)^{\frac{1}{2}} \vec{r} \cdot \vec{r} \quad \text{(norm formula)}
\]

\[
= \frac{\vec{r} \cdot \vec{v}}{|\vec{r}|} \quad \text{(norm formula)}
\]
6. (a) State the Triangle Inequality in both vector and scalar forms. (The scalar form should be expressed as an inequality for real numbers \(x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n\).)

\[
\sqrt{\sum_{i=1}^{n} (x_i + y_i)^2} \leq \sqrt{\sum_{i=1}^{n} x_i^2} + \sqrt{\sum_{i=1}^{n} y_i^2}
\]

\[
|\vec{x} + \vec{y}| \leq |\vec{x}| + |\vec{y}|
\]

(b) **Bonus question:** Derive the vector form of the Cauchy-Schwarz Inequality from the vector form of the Triangle Inequality. Be sure to use correct notation, and justify any key steps (e.g., "by properties of ...")

*Note that the problem specifically asks for a derivation of Cauchy-Schwarz from the Triangle Inequality and not by any other means. It does not ask for a proof of the Triangle Inequality. Thus, your argument should start out with the Triangle Inequality, and, via a chain of logical deductions (with each step followin g from the previous one), end with the Cauchy-Schwarz inequality. You can use properties of dot/cross products (e.g., distributivity), but say so if you do. Make sure to use correct notation, and clearly distinguish between vectors and scalars and between various types of multiplication such as dot product, cross product, scalar multiplication."

---

**Start with Triangle Inequality in vector form:**

\[
|\vec{x} + \vec{y}| \leq |\vec{x}| + |\vec{y}|
\]

**Square both sides:**

\[
|\vec{x} + \vec{y}|^2 \leq (|\vec{x}| + |\vec{y}|)^2
\]

**Use norm formula on left side:**

\[
(\vec{x} + \vec{y}) \cdot (\vec{x} + \vec{y}) \leq |\vec{x}|^2 + 2|\vec{x}| |\vec{y}| + |\vec{y}|^2
\]

**Distribute:**

\[
\vec{x} \cdot \vec{x} + 2 \vec{x} \cdot \vec{y} + \vec{y} \cdot \vec{y} \leq |\vec{x}|^2 + 2 |\vec{x}| |\vec{y}| + |\vec{y}|^2
\]

**Use norm formula:**

\[
|\vec{x}|^2 + 2 \vec{x} \cdot \vec{y} + |\vec{y}|^2 \leq |\vec{x}|^2 + 2 |\vec{x}| |\vec{y}| + |\vec{y}|^2
\]

**Same argument with -\vec{y}:**

1. \[
\frac{\vec{x} \cdot \vec{y}}{|\vec{x}|} \leq |\vec{y}|
\]
2. \[
-\vec{x} \cdot \vec{y} \leq |\vec{x}| |\vec{y}|
\]

**Cauchy-Schwarz**