Recap from Chapter 13: Curves in $\mathbb{R}^n$

- **Parametrization:** The standard way to specify a curve $C$ is via a parametrization $\mathbf{r}(t)$, a vector function that traces the curve as $t$ ranges over a parameter interval $a \leq t \leq b$. The vector function $\mathbf{r}(t)$ can be thought of as describing the position of a moving object at time $t$, and the curve $C$ as the path of motion between times $t = a$ and $t = b$. The component functions of $\mathbf{r}(t)$ are denoted by $x(t)$, $y(t)$, etc. Here are some useful parametrizations that you should know:

  - Line segment from $a$ to $b$ (in $\mathbb{R}^n$): $\mathbf{r}(t) = a + t(b - a)$, $0 \leq t \leq 1$.
  - Unit circle in the plane, in clockwise direction (in $\mathbb{R}^2$): $x(t) = \cos t$, $y(t) = \sin t$, $0 \leq t \leq 2\pi$.
  - Graph of function $y = f(x)$ from $x = a$ to $x = b$ (in $\mathbb{R}^2$): $x(t) = t$, $y(t) = f(t)$, $a \leq t \leq b$.

- **Technical notes:**

  - **Smooth and piecewise smooth curves:** A curve is called smooth if it has a parametrization $\mathbf{r}(t)$ whose derivative, $\mathbf{r}'(t)$, is continuous and not equal to 0. Intuitively, a smooth curve is one that does not have sharp corners. The curve is called piecewise smooth if it can be broken up into finitely many smooth curves. For example, the boundary curve of a triangle is piecewise smooth since each of the three sides that constitute the boundary is a smooth curve.

  - **Orientation:** The orientation of a curve denotes the direction in which it is traversed (e.g., from $A$ to $B$ versus from $B$ to $A$, or clockwise versus counterclockwise). Reversing the orientation of a curve $C$ changes a line integral over $C$ into its negative value, just like interchanging the limits in an ordinary one variable integral flips the sign of this integral. That is, we have $\int_C \cdots = -\int_{-C} \cdots$, where $-C$ denotes the curve $C$ with the orientation reversed and the integral is any of the various types of line integrals below.

### Line integral over scalar functions

- **Basic definition:**

  
  \[
  \int_C f \, ds = \int_a^b f(\mathbf{r}(t))|\mathbf{r}'(t)| \, dt \quad \text{(line integral w.r. to arclength)}
  \]

  \[
  \int_C f \, dx = \int_a^b f(\mathbf{r}(t))x'(t) \, dt \quad \text{(line integral w.r. to coordinate variable $x$)}
  \]

Integrals with respect to other coordinate variables are defined analogously. Note that to use the computational formulas all involve a parametrization $\mathbf{r}(t)$ of the curve $C$. **THUS, THE FIRST STEP IN COMPUTING LINE INTEGRALS SHOULD BE TO FIND A PARAMETRIZATION FOR $C$ (IF NOT ALREADY GIVEN).**

- **Interpretations and applications:**

  - **Arclength:** $L = \int_C 1 \, ds$ is the length of the curve $C$.
  - **Mass and center of mass of wire:** Consider a wire of shape $C$ with mass density function $\rho$. Then $m = \int_C \rho \, ds$ is the total mass of the wire, $\overline{x} = (1/m) \int_C x \rho \, ds$ and $\overline{y} = (1/m) \int_C y \rho \, ds$ are the $x$- and $y$-coordinates of the center of mass.
  - **Line integrals with respect to coordinate variables:** Integrals such as $\int_C xy \, dx$ represent line integrals that must not be confused with ordinary one variable integrals. The $C$ under the integral sign indicates that we are dealing with a line integral, rather than a one variable integral. **DO NOT TREAT SUCH INTEGRALS AS ORDINARY ONE VARIABLE INTEGRALS WITH RESPECT TO $X$ AND TRY TO SIMPLY INTEGRATE WITH RESPECT TO $X$.** The proper way to evaluate such an integral is by parametrizing the curve $C$ and converting the integral to an integral with respect to the parameter $t$ using the above formula.
Line integral over vector fields

A vector field in \( \mathbb{R}^n \) is a function \( \mathbf{F} \) that assigns to each point \( r \) in \( \mathbb{R}^n \) a vector \( \mathbf{F}(r) \) in \( \mathbb{R}^n \). (For example, \( \mathbf{F}(r) \) might represent a force vector acting at the point \( r \); see Section 16.1 for examples and pictures of vector fields in \( \mathbb{R}^2 \) and \( \mathbb{R}^3 \).) From a mathematical point of view, \( \mathbf{F} \) is simply a function from \( \mathbb{R}^n \) to \( \mathbb{R}^n \), with the input interpreted as a point in \( \mathbb{R}^n \) and the output interpreted as a vector in \( \mathbb{R}^n \).

- Basic definition:

\[
\int_C \mathbf{F} \cdot dr = \int_a^b \mathbf{F}(r(t)) \cdot r'(t) \, dt
\]

- Interpretations and applications: In physics, the line integral of \( \mathbf{F} \) over \( C \) represents the work performed when moving an object along the curve \( C \) under a force field \( \mathbf{F} \). In mathematics, the above type of line integral occurs in Green’s Theorem and in the Fundamental Theorem for Line Integrals.

Connection between vector and scalar line integrals

- Tangential component formula: Writing \( dr = r'(t)dt = T(t) \, |r'(t)|dt \), we have \( \mathbf{F} \cdot dr = \mathbf{F} \cdot T |r'(t)|dt = \mathbf{F} \cdot T \, ds \), giving the following relation:

\[
\int_C \mathbf{F} \cdot dr = \int_C \mathbf{F} \cdot T \, ds = \int_C F_T \, ds
\]

Geometrically, \( F_T = \mathbf{F} \cdot T \) is the projection of \( \mathbf{F} \) onto the tangential vector \( T \) for the curve, i.e., the tangential component of \( \mathbf{F} \) along the curve \( C \). Thus, the formula shows: The vector line integral of \( \mathbf{F} \) is the scalar line integral of the tangential component of \( \mathbf{F} \) along the curve.

- Coordinate formula: Assume \( \mathbf{F} \) is a vector field in \( \mathbb{R}^2 \), write \( \mathbf{F} = \langle P, Q \rangle \) and \( dr = \langle dx, dy \rangle \). Then formally \( \mathbf{F} \cdot dr = \langle P, Q \rangle \cdot \langle dx, dy \rangle = P \, dx + Q \, dy \), giving the following relation:

\[
\int_C \mathbf{F} \cdot dr = \int_C P \, dx + Q \, dy = \int_C P \, dx + \int_C Q \, dy
\]

- Differential elements: An easy way to remember the various formulas for line integrals is via the formulas for corresponding differential elements, i.e., the \( d \ldots \) expressions in the integrals: \( ds = |r'(t)| \, dt \) the scalar arclength element, \( dr = r'(t) \, dt \) the vector arclength element, and \( dx = x'(t) \, dt \), \( dy = y'(t) \, dt \), etc., the differentials of the coordinate variables.

The Fundamental Theorem for Line Integrals (FTLI)

In the special case when \( \mathbf{F} \) is a gradient field (or conservative field) i.e., if \( \mathbf{F} = \nabla f \) for some scalar field \( f \) (called potential), the line integral of \( \mathbf{F} \) for any curve \( C \) from \( A \) to \( B \) is simply the difference between the values of \( f \) at the end point \( B \) and start point \( A \) of the curve \( C \):

\[
\int_C \mathbf{F} \, dr = \int_C \nabla f \cdot dr = f(B) - f(A)
\]

Note that this formula can only be applied in the special case of vector fields \( \mathbf{F} \) that are conservative; in see Section 16.3 for more on this.