Written HW Assignment 9, DUE IN CLASS TUESDAY, 4/1/2014

Use this sheet as cover sheet and staple it to the assignment. Do the problems in order. Make sure that each problem is clearly labelled. Allow plenty of space for the problems. Write legibly, use proper mathematical notation, show all work, and provide explanations where appropriate (especially in problems asking you to “show” or “prove” something). You will get points off if your writing is poorly legible, or if you don’t provide adequate explanation. (If you are unsure what is considered adequate, just ask!)

HW 9 Problems

About this assignment: The first few problems (27, 28, and 29 from 15.5 and 51 from 15.7) are applications of double and triple integrals to probability. See Section 15.5, p. 1008–10010, and Examples 6–8, for the relevant terminology, definitions and formulas. The formulas are very much analogous to those for mass and center of mass: a probability density function behaves much like a mass density function, and the “expected values” for $x$ and $y$ ($x$-mean and $y$-mean) are like the $x$- and $y$-coordinates for the center of mass.

Note: In exams you have to be able to compute probabilities and expectations with given density functions $f(x, y)$ (as in the problems in this homework), but you don’t need to know formulas for specific densities such as the exponential or normal densities that occur in Examples 7 or 8.

The next group of problems is on triple integrals in cylindrical and spherical coordinates from Sections 15.8 and 15.9.

The final set of problems includes review problems on double and triple integrals taken from the review section at the end of Chapter 15 (p. 1050).

Triple integral problems can lead to tedious computations, but I tried to avoid those kinds of problems. The assigned problems require only a moderate amount of computations.

See the back of this page for tips on doing double and triple integrals.

1. 15.5:27
2. 15.5:28
3. 15.5:29
4. 15.7:51
5. 15.8:17
6. 15.8:28
7. 15.9:27
8. 15.9:35
9. 15.R:21
10. 15.R:28
11. 15.R:35(a)(b) (skip (c))
12. 15.R:54

*** Turn page for advice on doing double and triple integrals ***
Advice on doing double and triple integrals

- **ALWAYS SKETCH THE REGION OF INTEGRATION.** The region $R$ of integration is a region in the $xy$-plane that represents the “base” or “floor” of the object whose volume/mass/etc we want to compute.

In all but the simplest cases (e.g., rectangular regions), you should sketch $R$, then use the picture to “sweep out” the region either vertically or horizontally and “read off” the corresponding bounds for $x$ and $y$ from the picture. Don’t try to obtain the inequalities “in the abstract”, without a reference to the picture. This applies in particular to cases where the order of integration is reversed.

- **IF APPROPRIATE, ALSO SKETCH A VERTICAL CROSS SECTION.** For problems involving radially symmetric regions it is useful to sketch (in addition to the region $R$ which represents a horizontal layer), a vertical cross section of the region, i.e., a plot of $z$ against $r$ (or, equivalently, a cross-section with the $xz$-plane (i.e., $y = 0$).

- **Indicate the integration variable explicitly in integration limits.** For example, write

$$\int_0^1 \int_{y=x}^1 x y \, dx \, dy \quad \text{instead of} \quad \int_0^1 \int_x^1 x y \, dx \, dy.$$ 

The book doesn’t do this, but using such explicit notation helps cut down on mistakes. The same applies to the expressions obtained when carrying out the integrations. For example, for the inner integral above, write $[xy^2/2]_{y=0}$ rather than $[xy^2/2]_0$. In the latter form, it would be easy to make the mistake and substitute the wrong variable ($x$ instead of $y$).

- **Integration techniques:** You need to know (or review) standard integration techniques such as substitution, and integration by parts. Substitution is often needed for double integrals in polar coordinates (e.g., $\int (1 + r^2)^{-1} r \, dr$ or $\int r e^{-r^2} \, dr$).

- **Trig integrals:** Integrals in polar coordinates often lead to trig integrals. The most common such integrals are $\int \cos^2 x$ and $\int \sin^2 x$; to evaluate these, use the identities $\cos^2 x = (1/2)(1 + \cos(2x))$, $\sin^2 x = (1/2)(1 - \cos(2x))$.

(You need not worry about more esoteric trig integrals e.g., integrals over higher powers of sin, cos, sec, etc.).

- **Integrals over exponential functions.** Many probability applications involve integrals over negative exponential functions such as $e^{-3x}$. A common mistake is to not take into account the minus sign in the exponent, or to substitute the limits incorrectly. The indefinite integral of $e^{\pm x}$ is $(1/e)e^{\pm x}$, so

$$\int e^{-3x} = (-1/3)e^{-3x}, \quad \text{and} \quad \int_0^\infty e^{-3x} \, dx = (1/3) e^{-3x} \bigg|_{x=0}^{x=\infty}.$$ 

In the latter integral the contribution of the upper limit, $x = \infty$, is 0 since $e^{-3x} \to 0$ as $x \to \infty$, the contribution of the lower limit, $x = 0$, is $1/3$ so altogether we get $0 - (-1/3) = 1/3$. Mistakes in probability integrals are usually easy to spot, since a probability has to be between 0 and 1. If you get a negative answer, or an answer greater than 1, in a question asking for a probability, you made a mistake.

- **Using symmetry:** For many, but not all, problems asking for a center of mass, one can use symmetry arguments to determine some of the coordinates. The key questions to ask in each case are: (a) Does the region of integration change if the variables are interchanged (e.g., $x$ replaced by $z$, and $z$ replaced by $y$)? (b) Does the integrand change if the variables are interchanged? Depending on the answers to these questions, there may be symmetry with respect to one of the variables, but not with respect to other variables.

- **Last, but not least, remember the NUMBER ONE RULE for setting up multiple integrals: THE OUTER INTEGRAL IN A MULTIPLE INTEGRAL MUST HAVE CONSTANT LIMITS.** Integrals like $\int_0^1 \int_0^1 x y \, dx \, dy$ are nonsense.